Adv Comput Math (2012) 37:39–91
DOI 10.1007/s10444-011-9194-3

Multiple traces boundary integral formulation for Helmholtz transmission problems

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Received: 26 November 2010 / Accepted: 16 January 2011 /
Published online: 11 October 2011

Abstract We present a novel boundary integral formulation of the Helmholtz transmission problem for bounded composite scatterers (that is, piecewise constant material parameters in "subdomains") that directly lends itself to operator preconditioning via Calderón projectors. The method relies on local traces on subdomains and weak enforcement of transmission conditions. The variational formulation is set in Cartesian products of standard Dirichlet and special Neumann trace spaces for which restriction and extension by zero are well defined. In particular, the Neumann trace spaces over each subdomain boundary are built as piecewise $\widetilde{H}^{-1/2}$-distributions over each associated interface. Through the use of interior Calderón projectors, the problem is cast in variational Galerkin form with an operator matrix whose diagonal is composed of block boundary integral operators associated with the subdomains. We show existence and uniqueness of solutions based on an extension of Lions' projection lemma for non-closed subspaces. We also investigate asymptotic quasi-optimality of conforming boundary element Galerkin discretization. Numerical experiments in 2-D confirm the efficacy of the method and a performance matching that of another widely used boundary element discretization. They also demonstrate its amenability to different types of preconditioning.

Communicated by Charles Micchelli.

This work was funded by the ETH Fellowship "Boundary Element Methods for Multi-Dielectric Electromagnetic Scattering".

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Keywords Acoustic scattering · Boundary integral equations · Trace spaces · Calderón projectors · Boundary elements

Mathematics Subject Classifications (2010) 65N12 · 65N38 · 65R20

1 Introduction

We focus on the time-harmonic scattering of acoustic waves by a bounded penetrable object $\Omega \in \mathbb{R}^d$, $d = 2, 3$, composed of several subdomains Ω_i, $i = 1, \ldots, N$. Specifically, in each subdomain Ω_i the solution u satisfies a Helmholtz equation with wavenumber κ_i. This is generally referred to as a *Helmholtz Transmission Problem* (HTP) and is a relevant model for applications ranging from ultrasound and electromagnetic biomedical imaging [1, 51] to blood cell scattering [14] and antenna design [42]. A solution of the HTP on a given subdomain is related to the surrounding ones via continuity or *transmission conditions* for Dirichlet and Neumann traces across interfaces. More precisely, if u represents the total wave inside the scatterer Ω and the scattered field in the exterior $\Omega_0 := \mathbb{R}^d \setminus \bar{\Omega}$, the problem for N subdomains can be stated as follows:

Problem 1 (Multiple Transmission Problem) Seek u in a suitable functional space such that:

$$\begin{cases} -\Delta u - \kappa_i^2 u \;=\; 0 & \text{in } \Omega_i, \quad i = 0, \ldots, N, \\ +\text{inhom. transmission conditions} & \text{on } \partial\Omega, \\ +\text{homogeneous transmission conditions} & \text{on all interfaces } \Omega_i \cap \Omega_j, \\ +\text{radiation conditions} & \text{for } |\mathbf{x}| \longrightarrow \infty. \end{cases} \tag{1}$$

By employing adequate Green's functions, one can reduce the above transmission problem to boundary integral equations set on the subdomains' boundaries [8, 21, 23, 39]. In the case of a single homogeneous object ($N = 1$), direct boundary integral equations for the scattering transmission problem are readily deduced from the Calderón projector formulas. For scalar elliptic problems this is textbook knowledge, see [43, Sect. 3.4] and electromagnetic scattering at homogeneous objects is discussed in [39, Sect. 5.63] and [4, 15]. In a recent work by Laliena et al. [29], the authors propose a symmetric formulation based on [8] while introducing an additional mortar unknown. Unfortunately, this formulation is affected by spurious resonances and its extension to more than one subdomain is not clear. The HTP for one subdomain can also be converted into an intrinsically well-conditioned second-kind BIE but to our knowledge no extension to multiple subdomains is available and one must choose a first kind formulation.

There are different variants of boundary integral equations for several subdomains, i.e. $N > 1$. Loosely speaking, all these methods arise from the following variational equation:

$$a(u,v) \;:=\int_{\mathbb{R}^d} \mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}\mathbf{x}-\int_{\mathbb{R}^d}\kappa^2(\mathbf{x})u\,v\,\mathrm{d}\mathbf{x}=0,\quad \forall\, v\in H^1_{\mathrm{comp}}(\mathbb{R}^d), \tag{2}$$

where $\kappa^2(\mathbf{x})$ is simply defined as the piecewise constant function with values κ_i on each Ω_i, $i=0,\ldots,N$. Obviously,

$$a(u,v) \;=\sum_{i=0}^{N}\int_{\Omega_i}\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}\mathbf{x}-\sum_{i=0}^{N}\int_{\Omega_i}\kappa_i^2 uv\,\mathrm{d}\mathbf{x}\,. \tag{3}$$

Now, using Green's first formula over one subdomain and $\Delta u+\kappa_i^2 u=0$, we arrive at

$$\int_{\Omega_i}\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}\mathbf{x}-\int_{\Omega_i}\kappa_i^2 uv\,\mathrm{d}\mathbf{x}\;=\int_{\partial\Omega_i}\left(\mathbf{n}^i\cdot\mathbf{grad}\,u\right)\,v\,\mathrm{d}s\,, \tag{4}$$

and, thus, if $\mathbf{n}^i$ denotes the outward unit normal to Ω_i, (2) reduces to the sum

$$a(u,v)\;=\sum_{i=0}^{N}\int_{\partial\Omega_i}\left(\mathbf{n}^i\cdot\mathbf{grad}\,u\right)\,v\,ds\,, \tag{5}$$

wherein terms in brackets are the new unknowns and represent Neumann traces taken from the interior of Ω_i. Notice that the trace of v is in fact globally defined over the set of interfaces $\bigcup_i\partial\Omega_i$. At this point, one has two choices to describe the Neumann data:

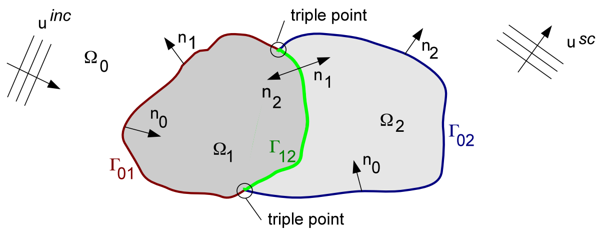
(i) as independent unknowns defined over each subdomain boundary $\partial\Omega_i$ and linked by continuity conditions—*local traces*,
(ii) as suitably oriented restrictions of an intrinsically defined Neumann trace over the entire set of interfaces—*global trace*.

In either case, Neumann traces are related to their Dirichlet counterparts via Dirichlet-to-Neumann (DtN) operators.

In [49], von Petersdorff presents a first-kind integral formulation based on DtN operators mapping a global Neumann trace defined over the entire set of interfaces and equipped with an intrinsic orientation. This so-called *Single Trace Formulation* (STF) requires local orientation operators according to Ω_i and has a minimum number of degrees of freedom (DOFs). Moreover, one can show existence and uniqueness of solutions and asymptotic quasi-optimality for Galerkin boundary element discretization.

Yet, as is typical of first-kind integral equations, the associated discrete linear problems on fine meshes suffer from poor conditioning. Unfortunately, standard preconditioning techniques are not successful, in particular *operator preconditioning* [5, 17, 33, 47], as there are no Calderón identities on global trace spaces due to the presence of triple points (2-D) or multiply shared edges (3-D) (see Fig. 1).

Fig. 1 Geometry, notation and normal orientations



Alternative boundary integral formulations based on local definitions for Neumann traces can be derived as non-overlapping domain decomposition methods [12] or substructuring methods [13, 20]. These are based on relating transmission conditions to Lagrange multipliers and reducing the model to its Schur complement. In [30], the foundations of the *boundary element tearing and interconnecting* (BETI) technique are laid following its finite element counterpart [10]. Since independent approximations are realized on each subdomain including its boundary, a discontinuous approximation space is retrieved. Then, the global continuity of the solution is enforced by point-wise algebraic constraints, modeled by Lagrange multipliers. The resulting saddle point problem is equivalent to a dual problem which is symmetric positive semidefinite. Although the method can readily be preconditioned, inversion of local discrete Steklov–Poincaré operators as well as local and global preconditioners for the dual problem [22] is required. Moreover, these methods are prone to spurious modes. A remedy for this consists in introducing Robin-type conditions instead of classic transmission conditions [48]. Nonetheless, the model still requires the inclusion of Lagrange multipliers.

In this work, we introduce a formulation in terms of both local Dirichlet and local Neumann subdomain boundary traces henceforth called *Multiple Traces Formulation* (MTF) which enjoys existence and uniqueness of solutions and is easy to precondition. Its construction employs

(i) Calderón projectors for the individual subdomains; and,
(ii) weakly enforced jump conditions across interfaces and partial subdomain boundaries.

This entails extending and restricting Neumann traces over the interfaces, and thus, instead of working on standard local Neumann basis on each subdomain boundary, we have to use piecewise $\widetilde{H}^{-1/2}$-distributions per interface. They supply valid test functions for a variational formulation, which forms the foundation for Galerkin discretization. This results in a system matrix whose block diagonal is composed of discrete boundary integral operators amenable to Calderón preconditioning.

The plan of the present article is as follows. Section 2 introduces the pertinent definitions for geometric parameters, functional space elements, trace and integral operators. In Section 3.1, we derive and analyze the MTF for the simple case of one subdomain ($N = 1$) and show existence and uniqueness of solutions for the method. Extension to many subdomains is given in

Section 3.2 wherein the uniqueness result is found via multiple applications of the representation formula and existence is based on an extensions of Lion's projection lemma for non-closed subspaces [31]. In Section 4 we analyze the convergence of a low-order boundary element Galerkin discretization. Numerical experiments in 2-D are given in Section 5 which validate the method for one and two subdomains when compared to other alternatives and we show its amenability to different types of preconditioning. A particular advantage of our proposed methodology relies on the straightforward implementation using standard BEM codes.

Remark 1 In algorithmic terms, the method outlined in the paper can be easily extended to more general Helmholtz transmission problems, where, with $\alpha_i > 0$, the solution u satisfies

$$-\operatorname{div}\alpha_i\,\mathbf{grad}\,u - \kappa_i^2 u \;=\; 0, \quad \text{in } \Omega_i, \quad i = 0, \ldots, N\,, \tag{6}$$

plus appropriate transmission and radiation conditions as in (1). Our theory can also be extended to this setting, though regularity estimates will become more delicate. For the sake of readability we forgo this generality.

2 Preliminaries

2.1 Geometry

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, designate the part of space occupied by a bounded object, assumed to be a curvilinear Lipschitz polygon or polyhedron, accordingly, with boundary $\partial\Omega$, composed of N subdomains Ω_i, i.e.

$$\bar{\Omega} \;=\; \bigcup_{i=1}^{N} \bar{\Omega}_i,$$

where the Ω_i are mutually disjoint, curvilinear Lipschitz polygons or polyhedra with boundaries $\partial\Omega_i$. Denote the exterior isotropic unbounded domain by $\Omega_0 := \mathbb{R}^d \setminus \bar{\Omega}$. For each Ω_i, we write its complement as $\Omega_i^c := \mathbb{R}^d \setminus \bar{\Omega}_i$.

Furthermore, let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$ represent the interface between domains Ω_i and Ω_j, equal to the empty set, if the domains are not adjacent. Notice that $\Gamma_{ij} = \Gamma_{ji}$ and that each $\partial\Omega_i$ can be decomposed into its interfaces:

$$\partial\Omega_i \;=\; \bigcup_{j\in\Lambda_i} \bar{\Gamma}_{ij}, \tag{7}$$

where we have introduced the index set:

$$\Lambda_i \;:=\; \left\{ j = 0, \ldots, N \;:\; j \neq i \quad \text{and} \quad \Gamma_{ij} \neq \emptyset \right\}. \tag{8}$$

The union of all interfaces Γ_{ij} or *skeleton* is denoted by Σ and we set $\Sigma_0 := \Sigma \setminus \partial\Omega$, representing the union of only interior interfaces. Lastly, denote by Υ the union of points (2-D) or curves (3-D) on Σ shared by more than two subdomains. This describes the set of triple points (2-D) or "wire-basket" (3-D) of Σ.

Remark 2 The geometric assumptions allow for non-connected boundaries $\partial\Omega_i$. Although this imposes no further difficulties in the following analysis, henceforth we assume all $\{\Omega_i\}_{i=1,N}$ homeomorphic to a disk (sphere) in $\mathbb{R}^2$ ($\mathbb{R}^3$) as depicted in Fig. 1.

2.2 Functional framework

Let D denote a d-dimensional manifold. We write $\mathscr{D}(D)$ for compactly supported infinitely differentiable functions. The space of distributions or linear functionals $\mathscr{D}(D)$ is $\mathscr{D}'(D)$ and $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions [2]. Duality products are denoted by angular brackets, $\langle\cdot\,,\,\cdot\rangle$, with subscripts accounting for specific duality pairings. Adjoint operators are denoted by a prime superscript. In the case of Hilbert spaces, the inner product is denoted by round brackets. Adjoint operators with respect to the inner product are designated by an asterisk. For complex-valued functions, one defines the sesquilinear form

$$(u\,,\,v)_{X'\times X} \;:=\; \langle u\,,\,\overline{v}\rangle_{X'\times X} \quad u\in X'\,, v\in X, \tag{9}$$

on the product space $X'\times X$.

For $s\in\mathbb{R}$, $H^s(\mathbb{R}^d)$ are the classic Sobolev spaces [34, Chapter 3] with $H^0(\mathbb{R}^d)\equiv L^2(\mathbb{R}^d)$ as pivot space. If P is a partial differential operator, one usually works in the following subspace of $H^s(\mathbb{R}^d)$:

$$H^s(\mathsf{P},\mathbb{R}^d) \;:=\; \left\{u\in H^s(\mathbb{R}^d)\;:\;\mathsf{P}u\in L^2(\mathbb{R}^d)\right\}. \tag{10}$$

with corresponding graph norm. For any non-empty open set $D\subseteq\mathbb{R}^d$, one writes

$$H^s(D) \;=\; \left\{u\in\mathscr{D}'(D)\;:\;u=U|_D \quad \text{for some } U\in H^s(\mathbb{R}^d)\,\right\} \tag{11}$$

and equivalently for $H^s(\mathsf{P},D)$. For $s\geq 0$, we say that a distribution belongs to the local Sobolev space $H^s_{\text{loc}}(\mathbb{R}^d)$ (resp. $H^s_{\text{loc}}(\mathsf{P},\mathbb{R}^d)$) if its restriction to every compact set $K\Subset\mathbb{R}^d$ lies in $H^s(K)$ (resp. $H^s(\mathsf{P},K)$). Also, we denote by $H^s_{\text{comp}}(D)$ the space of $H^s(D)$-functions compactly supported in D. If D has a boundary, we assume that it can be extended to a closed manifold $\widetilde{D}$, with $D\subset\widetilde{D}$, and write $\widetilde{u}$ for the extension of u by zero over $\widetilde{D}\setminus D$. For $s>0$ and D Lipschitz, one defines the closed subspace of $H^s(D)$:

$$\widetilde{H}^s(D) \;:=\; \left\{u\in H^s(D)\;:\;\widetilde{u}\in H^s(\widetilde{D})\right\}, \tag{12}$$

provided with the norm $\|u\|_{\widetilde{H}^s(D)}=\|\widetilde{u}\|_{H^s(\widetilde{D})}$, where the last norm is the standard one. If D is a bounded domain in $\mathbb{R}^d$ then one uses $\widetilde{D}:=\mathbb{R}^d$ and if D is closed $\widetilde{H}^s(D)\equiv H^s(D)$. For negative s, we identify $\widetilde{H}^s(D)$ with the dual space of $H^{-s}(D)$. In particular,

$$\widetilde{H}^{-1/2}(D) \;\equiv\; \left(H^{1/2}(D)\right)' \quad\text{and}\quad H^{-1/2}(D) \;\equiv\; \left(\widetilde{H}^{1/2}(D)\right)'. \tag{13}$$

2.2.1 Standard trace spaces and cross pairings

As shorthand, we define the product trace spaces over closed boundaries $\partial\Omega_i$:

$$\mathbf{V}_i^s \;:=\; H^{s+1/2}(\partial\Omega_i) \times H^{s-1/2}(\partial\Omega_i), \quad s \in \mathbb{R}, \tag{14}$$

equipped with the norm:

$$\|\cdot\|_{\mathbf{V}_i^s} = \|\cdot\|_{H^{s+1/2}(\partial\Omega_i)} + \|\cdot\|_{H^{s-1/2}(\partial\Omega_i)} \tag{15}$$

and set $\mathbf{V}_i \equiv \mathbf{V}_i^0$ as the standard Cauchy data space. Let Id_i denote the identity operator. If Q_i is the matrix operator that exchanges Dirichlet and Neumann data on a boundary $\partial\Omega_i$, i.e.

$$\mathsf{Q}_i := \begin{pmatrix} 0 & \mathsf{Id}_i \\ \mathsf{Id}_i & 0 \end{pmatrix} \;:\; \mathbf{V}_i \longrightarrow \mathbf{V}_i' \,, \tag{16}$$

we can define the dual of $\mathbf{V}_i$ as $\mathbf{V}_i' := \mathsf{Q}_i\,\mathbf{V}_i$ with duality product obtained as the sum of component-wise dual products. If one identifies $\mathbf{V}_i''$ with $\mathbf{V}_i$, one can conclude that $\mathsf{Q}_i' = \mathsf{Q}_i$, where the prime denotes adjoint in the standard dual sense. Thus, $\mathsf{Q}_i^2 = \mathsf{Id}_i$ and $\mathsf{Q}_i^{-1} = \mathsf{Q}_i$. For two elements $\mathbf{u}, \mathbf{v} \in \mathbf{V}_i$, we define their dual product as

$$\langle \mathbf{u}\,,\, \mathbf{v}\rangle_{\times,i} \;:=\; \left\langle \mathsf{Q}_i\,\mathbf{u}\,,\, \mathbf{v}\right\rangle_i \;=\; \left\langle \mathbf{u}\,,\, \mathsf{Q}_i\,\mathbf{v}\right\rangle_i , \tag{17}$$

and their associated sesquilinear form:

$$(\mathbf{u}\,,\, \mathbf{v})_{\times,i} \;:=\; \langle \mathbf{u}\,,\, \overline{\mathbf{v}}\rangle_{\times,i} \,. \tag{18}$$

In the forthcoming analysis, it will be convenient to introduce the $\times$*-adjoint* of an operator $\mathsf{H}_i : \mathbf{V}_i \to \mathbf{V}_i$, through the following relation:

$$\left(\mathsf{H}_i^\dagger\,\mathbf{u}\,,\, \mathbf{v}\right)_{\times,i} \;:=\; (\mathbf{u}\,,\, \mathsf{H}_i\,\mathbf{v})_{\times,i}\,, \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}_i \,. \tag{19}$$

Consequently, it holds $\mathsf{H}_i^\dagger = \mathsf{Q}_i\,\mathsf{H}_i'\,\mathsf{Q}_i : \mathbf{V}_i \to \mathbf{V}_i$.

2.2.2 Piecewise trace spaces

In the case of multiple interfaces, we will require the piecewise or broken spaces:

$$\widetilde{H}_{\mathrm{pw}}^{\mp 1/2}(\partial\Omega_i) \;:=\; \left\{u \in H^{\mp 1/2}(\partial\Omega_i) \;:\; u|_{\Gamma_{ij}} \in \widetilde{H}^{\mp 1/2}(\Gamma_{ij}), \quad \forall\, j \in \Lambda_i\right\}, \tag{20}$$

whose dual can be identified with

$$H_{\mathrm{pw}}^{\pm 1/2}(\partial\Omega_i) \;:=\; \left\{u \in \mathscr{D}'(\partial\Omega_i) \;:\; u|_{\Gamma_{ij}} \in H^{\pm 1/2}(\Gamma_{ij}), \quad \forall\, j \in \Lambda_i\right\}, \tag{21}$$

respectively. The following space inclusion chain with $L^2(\partial\Omega_i)$ as pivot space holds

$$\widetilde{H}_{\mathrm{pw}}^{1/2} \subset H^{1/2} \subset H_{\mathrm{pw}}^{1/2} \subset L^2 \subset \widetilde{H}_{\mathrm{pw}}^{-1/2} \subset H^{-1/2} \subset H_{\mathrm{pw}}^{-1/2}, \tag{22}$$

where the domain of definition, $\partial\Omega_i$, is implied. With this, one can build the product spaces

$$\begin{aligned} \mathbf{V}_{\mathrm{pw},i} &:= H^{1/2}_{\mathrm{pw}}(\partial\Omega_i) \times H^{-1/2}_{\mathrm{pw}}(\partial\Omega_i), \quad (23)\\ \widetilde{\mathbf{V}}_i &:= H^{1/2}(\partial\Omega_i) \times \widetilde{H}^{-1/2}_{\mathrm{pw}}(\partial\Omega_i), \quad (24)\\ \widetilde{\widetilde{\mathbf{V}}}_i &:= \widetilde{H}^{1/2}_{\mathrm{pw}}(\partial\Omega_i) \times \widetilde{H}^{-1/2}_{\mathrm{pw}}(\partial\Omega_i), \quad (25) \end{aligned}$$

for which we have the inclusions

$$\widetilde{\widetilde{\mathbf{V}}}_i \subset \widetilde{\mathbf{V}}_i \subset \mathbf{V}_i \subset \mathbf{V}_{\mathrm{pw},i}. \quad (26)$$

Note that these inclusions are strict, once $\partial\Omega_i$ is composed of more than one interface. Also observe that

$$\mathbf{V}_i' = \mathsf{Q}_i\,\mathbf{V}_i, \quad \mathbf{V}_{\mathrm{pw},i}' = \mathsf{Q}_i\,\widetilde{\widetilde{\mathbf{V}}}_i, \quad \text{but} \quad \widetilde{\mathbf{V}}_i' \subsetneq \mathsf{Q}_i\,\mathbf{V}_{\mathrm{pw},i}, \quad \widetilde{\mathbf{V}}_i' \neq \mathsf{Q}_i\,\widetilde{\mathbf{V}}_i. \quad (27)$$

We will also make use of the subspaces defined by restriction over Γ_{ij}:

$$\mathbf{V}_{\mathrm{pw},ij} := \mathbf{V}_{\mathrm{pw},i}\big|_{\Gamma_{ij}}, \quad \mathbf{V}_{ij} := \mathbf{V}_i\big|_{\Gamma_{ij}}, \quad \widetilde{\mathbf{V}}_{ij} := \widetilde{\mathbf{V}}_i\big|_{\Gamma_{ij}}, \quad \widetilde{\widetilde{\mathbf{V}}}_{ij} := \widetilde{\widetilde{\mathbf{V}}}_i\,\big|_{\Gamma_{ij}}, \quad (28)$$

with associated dual product $\langle \cdot\,,\,\cdot\rangle_{\times,ij}$ following (17). Notice that the following duality relations immediately follow:

$$\mathbf{V}_{ij}' = \mathsf{Q}_i\;\widetilde{\widetilde{\mathbf{V}}}_{ij} = \mathbf{V}_{\mathrm{pw},ij}' \quad \text{and} \quad \widetilde{\mathbf{V}}_{ij}' = \mathsf{Q}_i\,\widetilde{\mathbf{V}}_{ij}, \quad (29)$$

where the last one is not immediate from (27).

Remark 3 In [7], it was shown that C^∞-functions supported away from Υ are dense in the function spaces $H^1(\mathbb{R}^3)$. Thus, L^2-functions compactly supported inside interfaces Γ_{ij} will be dense in the Dirichlet trace spaces. Then, by duality one can show the dense embeddings:

$$\widetilde{\widetilde{\mathbf{V}}}_i \hookrightarrow \widetilde{\mathbf{V}}_i \hookrightarrow \mathbf{V}_i.$$

These embeddings are pivotal for the multiple traces Galerkin formulation for the HTP.

Remark 4 Recall that if the interface Γ_{ij} is a closed manifold, $\widetilde{H}^{-1/2}(\Gamma_{ij}) \equiv H^{-1/2}(\Gamma_{ij})$. Since we have assumed all Ω_i homeomorphic to the d-dimensional sphere, this case will only occur for $N = 1$.

2.2.3 Trace operators and transmission conditions

Let γ^i denote the trace operator taken from within Ω_i which maps $\mathscr{D}(\bar{\Omega}_i)$ onto $C^0(\partial\Omega_i)$. Its extension to Sobolev spaces is achieved by density [32]. Designate by $\mathbf{n}^i = [n^i_j]_{j=1}^d$ the unit outward normal to Ω_i. For clarity, we will distinguish between Dirichlet and Neumann traces defined as [43, Section 2.6], [34, Theorem 3.37],

$$\gamma_{\mathrm{D}}^i u := \gamma^i u \quad \text{and} \quad \gamma_{\mathrm{N}}^i u := \gamma^i \left(\mathbf{n}^i \cdot \mathbf{grad}\, u\right),$$

respectively, and construct the vector trace operator:

$$\gamma^i u := \begin{pmatrix} \gamma_{\mathrm{D}}^i u \\ \gamma_{\mathrm{N}}^i u \end{pmatrix} : H^1_{\mathrm{loc}}(\Delta, \Omega_i) \longrightarrow \mathbf{V}_i.$$

Now, when considering traces taken from the complementary domain Ω_i^c, denoted $\gamma^{i,c}$, it is customary to define the Neumann trace using the same sense of the outward normal to Ω_i. Thus, if $\Omega_j \subseteq \Omega_i^c$ the following transmission conditions hold in the sense of distributions:

$$\gamma_{\mathrm{D}}^j u\big|_{\Gamma_{ij}} = \gamma_{\mathrm{D}}^{i,c} u\big|_{\Gamma_{ij}} \quad \text{and} \quad \gamma_{\mathrm{N}}^j u\big|_{\Gamma_{ij}} = -\gamma_{\mathrm{N}}^{i,c} u\big|_{\Gamma_{ij}}, \quad u \in H^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d), \tag{30}$$

due to opposite senses of the normal vectors $\mathbf{n}^i$ and $\mathbf{n}^j$. We introduce the endomorphism on product spaces $\mathbf{V}_i$:

$$\mathsf{X}_i := \begin{pmatrix} \mathsf{Id}_i & 0 \\ 0 & -\mathsf{Id}_i \end{pmatrix} : \mathbf{V}_i \longrightarrow \mathbf{V}_i \tag{31}$$

to succinctly rewrite (30) as

$$\gamma^{i,c} u\big|_{\Gamma_{ij}} = \left(\mathsf{X}_j \gamma^j\right) u\big|_{\Gamma_{ij}}, \quad u \in H^1_{\mathrm{loc}}(\mathsf{P}, \mathbb{R}^d). \tag{32}$$

The action of X_i should be interpreted according to the associated trace operator. When there is no risk of confusion we will drop the subindex i. Moreover, one can easily show:

Lemma 1 *It holds $\mathsf{X}_i^2 = \mathsf{Id}_i$ and $\mathsf{X}_i^\dagger = -\mathsf{X}_i$ where $\mathsf{X}_i^\dagger$ is the $\times$-adjoint of X_i.*

Proof The first property is immediate. For the second one, take $\boldsymbol{\eta}^i$ and φ^i in $\mathbf{V}_i$. The $\times$-adjoint operator $\mathsf{X}_i^\dagger$ is defined through the equality $\left(\mathsf{X}_i^\dagger \boldsymbol{\eta}^i, \varphi^i\right)_{\times,i} = \left(\boldsymbol{\eta}^i, \mathsf{X}_i \varphi^i\right)_{\times,i}$. On the other hand, expansion of the duality pairing gives

$$\left(\boldsymbol{\eta}^i, \mathsf{X}_i \varphi^i\right)_{\times,i} = -\left(\eta_{\mathrm{D}}^i, \varphi_{\mathrm{N}}^i\right)_i + \left(\eta_{\mathrm{N}}^i, \varphi_{\mathrm{D}}^i\right)_i = -\left(\mathsf{X}_i \boldsymbol{\eta}^i, \varphi^i\right)_{\times,i}, \tag{33}$$

and the conclusion follows. □

2.2.4 Trace jumps and averages

Introduce the *trace jump operator*, $[\gamma]_{\partial\Omega_i}$, across the subdomain boundary $\partial\Omega_i$ as the standard exterior minus interior traces, i.e.

$$[\gamma]_{\partial\Omega_i} := \begin{pmatrix} [\gamma_{\mathrm{D}}]_{\partial\Omega_i} \\ [\gamma_{\mathrm{N}}]_{\partial\Omega_i} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathrm{D}}^{i,c} - \gamma_{\mathrm{D}}^i \\ \gamma_{\mathrm{N}}^{i,c} - \gamma_{\mathrm{N}}^i \end{pmatrix} \quad \text{on } \partial\Omega_i. \tag{34}$$

When restricting the above over an interface Γ_{ij}, the action of the jump operator over $u \in H^1_{\text{loc}}(\Delta, \mathbb{R}^d)$ takes the form:

$$\begin{aligned} [\gamma u]_{\Gamma_{ij}} &= \begin{pmatrix} \gamma_{\mathrm{D}}^{i,c} u\big|_{\Gamma_{ij}} - \gamma_{\mathrm{D}}^{i} u|_{\Gamma_{ij}} \\ \gamma_{\mathrm{N}}^{i,c} u\big|_{\Gamma_{ij}} - \gamma_{\mathrm{N}}^{i} u|_{\Gamma_{ij}} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathrm{D}}^{j} u\big|_{\Gamma_{ij}} - \gamma_{\mathrm{D}}^{i} u|_{\Gamma_{ij}} \\ -\gamma_{\mathrm{N}}^{j} u\big|_{\Gamma_{ij}} - \gamma_{\mathrm{N}}^{i} u\big|_{\Gamma_{ij}} \end{pmatrix} \\ &= \left(\mathsf{X}_j \gamma^j u - \gamma^i u\right)\big|_{\Gamma_{ij}} \end{aligned} \tag{35}$$

in distributional sense. Equivalently, we define *trace average operators* over $\partial\Omega_i$:

$$\{\gamma\}_{\partial\Omega_i} := \begin{pmatrix} \{\gamma_{\mathrm{D}}\}_{\partial\Omega_i} \\ \{\gamma_{\mathrm{N}}\}_{\partial\Omega_i} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \gamma_{\mathrm{D}}^{i,c} + \gamma_{\mathrm{D}}^{i} \\ \gamma_{\mathrm{N}}^{i,c} + \gamma_{\mathrm{N}}^{i} \end{pmatrix}, \tag{36}$$

whose restriction over Γ_{ij} is written down as

$$\{\gamma u\}_{\Gamma_{ij}} = \frac{1}{2} \begin{pmatrix} \gamma_{\mathrm{D}}^{j} u\big|_{\Gamma_{ij}} + \gamma_{\mathrm{D}}^{i} u\big|_{\Gamma_{ij}} \\ -\gamma_{\mathrm{N}}^{j} u\big|_{\Gamma_{ij}} + \gamma_{\mathrm{N}}^{i} u\big|_{\Gamma_{ij}} \end{pmatrix} = \frac{1}{2}\left(\mathsf{X}_j \gamma^j u + \gamma^i u\right)\big|_{\Gamma_{ij}}. \tag{37}$$

Remark 5 These definitions, and in particular sign conventions, are key when defining the boundary integral operators of Section 2.3.1.

2.2.5 Skeleton spaces

Let us also define Dirichlet and Neumann spaces over the skeleton Σ as follows. Let

$$\mathbb{H}_N^{\pm 1/2} := H^{\pm 1/2}(\partial\Omega_0) \times \cdots \times H^{\pm 1/2}(\partial\Omega_N),$$

then

$$H^{1/2}(\Sigma) := \left\{ \begin{array}{l} u \in \mathbb{H}_N^{1/2} : \exists\, U \in H^1(\mathbb{R}^d) \text{ s.t.} \\ \gamma_{\mathrm{D}}^i U = u|_{\partial\Omega_i} \in H^{1/2}(\partial\Omega_i),\ i = 0, \ldots, N, \end{array} \right. \subset \mathbb{H}_N^{1/2},$$

endowed with the norm:

$$\|u\|_{H^{1/2}(\Sigma)} := \inf\left\{\|U\|_{H^1(\mathbb{R}^3)} : U|_\Sigma = u\right\}.$$

As seen in (4), Neumann traces arise from integration by parts formulae. This invokes an *induced* or *relative orientation* of the boundary which is, in fact, given by the exterior normal $\mathbf{n}^i$ to subdomain Ω_i. Thus, local orientations are required to define *Neumann trace spaces*. Specifically,

$$H^{-1/2}(\Sigma) := \left\{ \psi \in \mathbb{H}_N^{-1/2} : \sum_i^N \langle \psi^i, v^i \rangle_{\partial\Omega_i} = 0, \quad \forall v \in H^{1/2}(\Sigma) \right\} \subset \mathbb{H}_N^{-1/2},$$

wherein ψ^i has been accordingly oriented (cf. (121)) and provided with the natural norm. These spaces constitute the intrinsic setting for the single trace formulation.

2.3 Integral operators

From this section on, we introduce the necessary elements for analyzing Helmholtz and Laplace transmission problems in the setting of boundary integral equations [43, Chapter 3], [34, Chapters 7–9]. Consider a single domain Ω_i with complement Ω_i^c and boundary $\partial\Omega_i$. One defines the Helmholtz operator $\mathsf{P}_i := -(\Delta + \kappa_i^2)$ over $\mathbb{R}^d \setminus \partial\Omega_i$ wherein κ_i is a real non-negative bounded constant over $\mathbb{R}^d \setminus \partial\Omega_i$–if $\kappa_i = 0$ one retrieves the Laplace operator $-\Delta$. We aim at solving the following problem:

Problem 2 Let $\Omega_i \subset \mathbb{R}^d$ and f have compact support in $\mathbb{R}^d \setminus \partial\Omega_i$. We seek $u \in H^1_{\text{loc}}(\Omega_i \cup \Omega_i^c)$ such that

$$\begin{cases} \mathsf{P}_i u \;=\; f, & \text{in } \Omega_i \cup \Omega_i^c, \\ +\text{radiation/decay conditions.} \end{cases} \tag{38}$$

The associated radiation conditions for $\kappa_i \neq 0$ assume a harmonic time dependence of the form $\exp(-i\omega t)$, where the pulsation ω is real and positive so that $\kappa_i^2 = \omega^2/c_i^2$ where c_i is the wave speed in Ω_i. For bounded scatterers, these conditions are due to Sommerfeld [34, 45]. We thus speak of solutions of Problem 2 as *Helmholtz radiating solutions*. In the static case ($\kappa_i = 0$), the conditions are written differently.

Let $G^i(\mathbf{x} - \mathbf{y}) \in \mathscr{S}'(\mathbb{R}^d)$ denote the fundamental solution for Helmholtz ($\kappa_i > 0$) and Laplacian equation ($\kappa_i = 0$). With it, the *Newton potential* for compactly supported functions f is [43, Section 3.1.1]

$$\mathcal{N}^i(f)(\mathbf{x}) \;:=\; \int_{\mathbb{R}^d} G^i(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d. \tag{39}$$

Similarly, one defines the standard single and double layer potentials:

$$\Psi^i_{\text{SL}} \;:=\; \mathcal{N}^i \circ \gamma_{\text{D}}^* \quad \text{and} \quad \Psi^i_{\text{DL}} \;:=\; \mathcal{N}^i \circ \gamma_{\text{N}}^*, \tag{40}$$

respectively, where the asterisk denotes adjoint traces with respect to the L^2-duality product taken on $\partial\Omega_i$.

Theorem 1 (Representation formula [43, 49], Section 3.1.1) *Let u be a solution of Problem 2 and suppose that $f := (\mathsf{P}_i u)|_{\mathbb{R}^d \setminus \partial\Omega_i}$ has compact support. Then, we have the integral representation:*

$$u(\mathbf{x}) \;=\; \mathcal{N}^i(f)(\mathbf{x}) \;+\; \Psi^i\left(\left[\gamma u\right]_{\partial\Omega_i}\right)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \partial\Omega_i, \tag{41}$$

where

$$\Psi^i\left(\left[\gamma u\right]_{\partial\Omega_i}\right) \;:=\; \Psi^i_{DL}\left(\left[\gamma_D u\right]_{\partial\Omega_i}\right) \;-\; \Psi^i_{SL}\left(\left[\gamma_N u\right]_{\partial\Omega_i}\right), \tag{42}$$

with trace jumps defined as in Section 2.2.4.

Remark 6 If $u|_{\Omega_i} \in H^1_{\mathrm{loc}}(\Omega_i)$ is a solution of the homogeneous problem $\mathsf{P}_i\, u = 0$ satisfying outgoing radiation conditions, then application of Theorem 1 with $f = 0$ and $u|_{\Omega_i^c} = 0$ gives

$$u \;=\; -\,\Psi^i_{\mathrm{DL}}\left(\gamma^i_{\mathrm{D}} u\right) \;+\; \Psi^i_{\mathrm{SL}}\left(\gamma^i_{\mathrm{N}} u\right) \quad \text{in } \mathbb{R}^d \setminus \partial\Omega_i\,. \tag{43}$$

For a solution $u \in H^1_{\mathrm{loc}}(\Omega_i^c)$ with $u|_{\Omega_i} = 0$ and $f = 0$, then

$$u \;=\; \Psi^i_{\mathrm{DL}}\left(\gamma^{i,c}_{\mathrm{D}} u\right) \;-\; \Psi^i_{\mathrm{SL}}\left(\gamma^{i,c}_{\mathrm{N}} u\right) \quad \text{in } \mathbb{R}^d \setminus \partial\Omega_i\,. \tag{44}$$

One can build a radiating solution for Problem 2 for any $\boldsymbol{\lambda}^i = (\lambda^i_{\mathrm{D}}, \lambda^i_{\mathrm{N}}) \in \mathbf{V}_i$ as

$$\Psi^i(\boldsymbol{\lambda}^i) \;=\; \Psi^i_{\mathrm{DL}}\left(\lambda^i_{\mathrm{D}}\right) \;-\; \Psi^i_{\mathrm{SL}}\left(\lambda^i_{\mathrm{N}}\right). \tag{45}$$

The above results will be extensively used in the derivation of the so-called Calderón projectors and in subsequent proofs.

2.3.1 Boundary integral operators

For each κ_i, let us introduce the standard boundary integral operators (BIOs) over $\partial\Omega_i$ [6, Theorem 3.1.16], [43, Section 3.1]:

$$\mathsf{V}_i \;:=\; \left\{\gamma_{\mathrm{D}}\, \Psi^i_{\mathrm{SL}}\right\}_{\partial\Omega_i} : H^{-1/2}(\partial\Omega_i) \;\longrightarrow\; H^{1/2}(\partial\Omega_i), \tag{46a}$$

$$\mathsf{K}'_i \;:=\; \left\{\gamma_{\mathrm{N}}\, \Psi^i_{\mathrm{SL}}\right\}_{\partial\Omega_i} : H^{-1/2}(\partial\Omega_i) \;\longrightarrow\; H^{-1/2}(\partial\Omega_i), \tag{46b}$$

$$\mathsf{K}_i \;:=\; \left\{\gamma_{\mathrm{D}}\, \Psi^i_{\mathrm{DL}}\right\}_{\partial\Omega_i} : H^{1/2}(\partial\Omega_i) \;\longrightarrow\; H^{1/2}(\partial\Omega_i), \tag{46c}$$

$$\mathsf{W}_i \;:=\; -\left\{\gamma_{\mathrm{N}}\, \Psi^i_{\mathrm{DL}}\right\}_{\partial\Omega_i} : H^{1/2}(\partial\Omega_i) \;\longrightarrow\; H^{-1/2}(\partial\Omega_i). \tag{46d}$$

For the potentials one can deduce [43, Section 3.3.1]

$$\gamma^{i,c}_{\mathrm{D}}\, \Psi^i_{\mathrm{SL}} \;=\; \mathsf{V}_i, \qquad \gamma^{i}_{\mathrm{D}}\, \Psi^i_{\mathrm{SL}} \;=\; \mathsf{V}_i, \tag{47a}$$

$$\gamma^{i,c}_{\mathrm{N}}\, \Psi^i_{\mathrm{SL}} \;=\; \frac{1}{2}\,\mathsf{Id} + \mathsf{K}'_i, \qquad \gamma^{i}_{\mathrm{N}}\, \Psi^i_{\mathrm{SL}} \;=\; -\frac{1}{2}\,\mathsf{Id} + \mathsf{K}'_i, \tag{47b}$$

$$\gamma^{i,c}_{\mathrm{D}}\, \Psi^i_{\mathrm{DL}} \;=\; -\frac{1}{2}\,\mathsf{Id} + \mathsf{K}_i, \qquad \gamma^{i}_{\mathrm{D}}\, \Psi^i_{\mathrm{DL}} \;=\; \frac{1}{2}\,\mathsf{Id} + \mathsf{K}_i, \tag{47c}$$

$$\gamma^{i,c}_{\mathrm{N}}\, \Psi^i_{\mathrm{DL}} \;=\; -\,\mathsf{W}_i, \qquad \gamma^{i}_{\mathrm{N}}\, \Psi^i_{\mathrm{DL}} \;=\; -\,\mathsf{W}_i\,. \tag{47d}$$

Remark 7 The reader should be aware of the varying definitions of W_i together with trace jump conventions when comparing different works [34, 39, 43]. Also, factors $\pm 1/2$ multiplying Id_i come classically only for smooth boundaries and should be modified according to the solid angle described at the point where the trace is taken. However, since all our domains are assumed to be piecewise smooth, the change of values only occurs in sets of zero measure and can be discarded.

2.3.2 Coercivity properties

Lemma 2 ([3, 43], Lemma 3.9.8) *If $\kappa_i \neq 0$, the following operators are compact*

$$\delta\,\mathsf{V}_i := \mathsf{V}_i - \mathsf{V}_0 \quad : \quad H^{-1/2}(\partial\Omega_i) \longrightarrow H^{1/2}(\partial\Omega_i), \tag{48a}$$

$$\delta\,\mathsf{K}'_i := \mathsf{K}'_i - \mathsf{K}'_0 \quad : \quad H^{-1/2}(\partial\Omega_i) \longrightarrow H^{-1/2}(\partial\Omega_i), \tag{48b}$$

$$\delta\,\mathsf{K}_i := \mathsf{K}_i - \mathsf{K}_0 \quad : \quad H^{1/2}(\partial\Omega_i) \longrightarrow H^{1/2}(\partial\Omega_i), \tag{48c}$$

$$\delta\,\mathsf{W}_i := \mathsf{W}_i - \mathsf{W}_0 \quad : \quad H^{1/2}(\partial\Omega_i) \longrightarrow H^{-1/2}(\partial\Omega_i). \tag{48d}$$

Lemma 3 ([43], Proposition 3.5.5) *Operators W_i and V_i satisfy Gårding-type inequalities:*

$$\mathfrak{Re}\left\{\left(\varphi\,,\ \left(\mathsf{V}_i + \mathsf{T}_{\mathsf{V}_i}\right)\varphi\right)_i\right\} \ \geq\ \alpha_{\mathsf{V}_i}\,\|\varphi\|^2_{H^{-1/2}(\partial\Omega_i)}\,, \qquad \forall\,\varphi \in H^{-1/2}(\partial\Omega_i), \tag{49}$$

$$\mathfrak{Re}\left\{\left(v\,,\ \left(\mathsf{W}_i + \mathsf{T}_{\mathsf{W}_i}\right)v\right)_i\right\} \ \geq\ \alpha_{\mathsf{W}_i}\,\|v\|^2_{H^{1/2}(\partial\Omega_i)}\,, \qquad \forall\,v \in H^{1/2}(\partial\Omega_i), \tag{50}$$

with $\alpha_{\mathsf{V}_i}, \alpha_{\mathsf{W}_i} > 0$ and $\mathsf{T}_{\mathsf{V}_i} : H^{-1/2}(\partial\Omega_i) \to H^{1/2}(\partial\Omega_i)$ and $\mathsf{T}_{\mathsf{W}_i} : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$ are compact. In particular, we can identify $\mathsf{T}_{\mathsf{V}_i} \equiv \delta\,\mathsf{V}_i$ and $\mathsf{T}_{\mathsf{W}_i} \equiv \delta\,\mathsf{W}_i$.

Lemma 4 ([19], Lemma 3.3) *Let $\partial\Omega_i$ be a Lipschitz boundary. There exists a compact operator $\mathsf{T}_{\mathsf{K}_i} : H^{-1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$ such that*

$$\left(\mathsf{K}'_i\,\varphi\,,\ v\right)_i \ =\ \left(\left(\mathsf{K}^*_i - \mathsf{T}_{\mathsf{K}_i}\right)\varphi\,,\ v\right)_i \tag{51}$$

holds true for all $\varphi \in H^{-1/2}(\partial\Omega_i)$ and $v \in H^{1/2}(\partial\Omega_i)$, where K^*_i is the L^2-adjoint of K_i.

Later on we will appeal to the smoothing properties of the above operators. These are summarized in the following lemma:

Lemma 5 ([36], Section 2, [46], Section 6.1, [43], Theorem 3.5.5) *If $\partial\Omega_i$ is piecewise smooth, the compact operators $\mathsf{T}_{\mathsf{V}_i}$, $\mathsf{T}_{\mathsf{W}_i}$ and $\mathsf{T}_{\mathsf{K}_i}$ defined in Lemmas 3 and 4 are smoothing, i.e.*

$$\mathsf{T}_{\mathsf{V}_i} \ :\quad H^{-1/2}(\partial\Omega_i) \longrightarrow H^{1}(\partial\Omega_i), \tag{52a}$$

$$\mathsf{T}_{\mathsf{K}_i} \ :\quad H^{-1/2}(\partial\Omega_i) \longrightarrow H^{1}(\partial\Omega_i), \tag{52b}$$

$$\mathsf{T}_{\mathsf{W}_i} \ :\quad H^{1/2}(\partial\Omega_i) \longrightarrow L^{2}(\partial\Omega_i). \tag{52c}$$

Proof We start with the proof for $\mathsf{T}_{\mathsf{V}_i}$. Based on the identifications of Lemma 3, and the definitions of the boundary operator and potentials (46a), (40), we analyze the mapping properties of the difference operators $\gamma_{\mathrm{D}}(\mathcal{N}^i - \mathcal{N}^0)\gamma^*_{\mathrm{D}}$. We first observe that the continuity of $\gamma_{\mathrm{D}} : H^1_{\mathrm{loc}}(\mathbb{R}^d) \to H^{1/2}(\partial\Omega_i)$ implies the continuity of the adjoint operator $\gamma^*_{\mathrm{D}} : H^{-1/2}(\partial\Omega_i) \to H^{-1}_{\mathrm{comp}}(\mathbb{R}^d)$. Secondly, it is well known [43, Remark 3.1.3] that $\mathcal{N}^i - \mathcal{N}^0 :$

$H^l_{\text{comp}}(\mathbb{R}^d) \to H^{l+4}_{\text{loc}}(\mathbb{R}^d)$, for $l \in \mathbb{R}$. Lastly, for piecewise smooth domains [34, Theorem 3.37], $\gamma_D : H^s(\mathbb{R}^d) \to H^{s-1/2}(\partial\Omega_i)$ for $1/2 < s \leq 3/2$. Thus, we have

$$H^{-1/2}(\partial\Omega_i) \underset{\gamma_D^*}{\longrightarrow} H^{-1}_{\text{comp}}(\mathbb{R}^d) \xrightarrow[\mathcal{N}^i - \mathcal{N}^0]{} H^3_{\text{loc}}(\mathbb{R}^d) \hookrightarrow H^{3/2}_{\text{loc}}(\mathbb{R}^d) \underset{\gamma_D}{\longrightarrow} H^1(\partial\Omega_i) \quad (53)$$

as stated in (52a), and where the curved arrow denotes compact injection (see [43, Theorem 2.6.7]). For $\mathsf{T}_{\mathsf{W}_i}$ and $\mathsf{T}_{\mathsf{K}_i}$, the proof follows similarly (see the proof of Lemma 3.9.8 in [43]). □

2.3.3 Calderón projectors

For a single κ_i, we recall the interior and exterior Calderón projectors acting on $\mathbf{V}_i$ denoted C_i and C_i^c, respectively [43, Section 3.6]. These are obtained by taking interior γ^i (resp. exterior $\gamma^{i,c}$) traces of the integral representations (43) (resp. (44)) and using (47)

$$\mathsf{C}_i := \frac{1}{2}\mathsf{Id} + \mathsf{A}_i \quad \text{and} \quad \mathsf{C}_i^c := \frac{1}{2}\mathsf{Id} - \mathsf{A}_i, \tag{54}$$

where

$$\mathsf{A}_i := \begin{pmatrix} -\mathsf{K}_i & \mathsf{V}_i \\ \mathsf{W}_i & \mathsf{K}_i' \end{pmatrix} : \mathbf{V}_i \longrightarrow \mathbf{V}_i. \tag{55}$$

They possess the projector property:

$$\mathsf{C}_i^2 = \mathsf{C}_i\,, \quad (\mathsf{C}_i^c)^2 = \mathsf{C}_i^c\,, \quad \text{and} \quad \mathsf{C}_i + \mathsf{C}_i^c = \mathsf{Id}_i, \tag{56}$$

which implies

$$\mathsf{A}_i^2 = \frac{1}{4}\mathsf{Id}_i\,. \tag{57}$$

A key result is the following characterization of possible traces for Helmholtz solution, see [19, Thm. 4.1]

Theorem 2 *If and only if $\boldsymbol{\lambda} \in \mathbf{V}_i$ satisfies $\mathsf{C}_i\boldsymbol{\lambda} = \boldsymbol{\lambda}$, there is a (radiating) Helmholtz solution u in Ω_i such that $\boldsymbol{\lambda} = \gamma^i u$.*

Theorem 3 [18] *The operator $\mathsf{A}_i : \mathbf{V}_i \to \mathbf{V}_i$ is coercive, i.e. it satisfies the Gårding inequality*

$$\mathfrak{Re}\left\{\left(\varphi\,,\ \left(\mathsf{A}_i + \mathsf{T}_{\mathsf{A}_i}\right)\varphi\right)_{\times,i}\right\} \geq \alpha_{\mathsf{A}_i}\,\|\varphi\|^2_{\mathbf{V}_i}\,, \quad \forall\,\varphi \in \mathbf{V}_i, \tag{58}$$

where $\mathsf{T}_{\mathsf{A}_i} : \mathbf{V}_i \to \mathbf{V}_i$ is a compact operator given by

$$\mathsf{T}_{\mathsf{A}_i} := \begin{pmatrix} \mathsf{T}_{\mathsf{K}_i} & \mathsf{T}_{\mathsf{V}_i} \\ \mathsf{T}_{\mathsf{W}_i} & 0 \end{pmatrix}, \tag{59}$$

with $\mathsf{T}_{\mathsf{V}_i} : H^{-1/2}(\partial\Omega_i) \to H^{1/2}(\partial\Omega_i)$, $\mathsf{T}_{\mathsf{W}_i} : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$, $\mathsf{T}_{\mathsf{K}_i} : H^{-1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$ compact themselves.

Lemma 6 *The operator* $\mathsf{T}_{\mathsf{A}_i}$ *is also smoothing, i.e. it maps* $\mathbf{V}_i$ *to* $\mathbf{V}_i^{1/2} = H^1(\partial\Omega_i) \times L^2(\partial\Omega_i)$.

Proof After observing (59), this is a direct consequence of Lemma 5. □

3 Multiple traces boundary integral equations

The gist of the MTF construction can be more easily understood when considering a single subdomain ($N = 1$). Thus, we first explain the derivation in this simple case and compare it to the single trace formulation (STF). Then, in Section 3.2, we present the full extension of the method to $N > 1$ subdomains highlighting the arising technical difficulties.

3.1 Single scatterer ($N = 1$)

Let Ω_1 be bounded, $\Omega_0 = \mathbb{R}^d \setminus \overline{\Omega}_1$ with interface $\Gamma = \Gamma_{10} = \partial\Omega$. Also, let $\mathbf{g} = (g_{\mathrm{D}}, g_{\mathrm{N}}) \in \mathbf{V}_0$. We seek $u \in H^1_{\mathrm{loc}}(\Omega_0 \cup \Omega_1)$ satisfying the Helmholtz transmission problem:

$$\begin{cases} \mathsf{P}_i\, u \;=\; 0, & \text{in } \Omega_i, \quad i = 0, 1, \\ [\gamma u] \;=\; \mathbf{g}, & \text{on } \Gamma, \\ + \text{ radiation conditions} & \text{for } |\mathbf{x}| \longrightarrow \infty, \end{cases} \tag{60}$$

where $\mathsf{P}_i := -(\Delta + \kappa_i^2)$ is defined over Ω_i with κ_i corresponding wavenumbers such that $\kappa_0 \neq \kappa_1$. As shown in Section 2.2.4, transmission conditions across the interface can be expressed as

$$\mathsf{X}_0\, \gamma^0 u \;=\; \gamma^1 u + \mathbf{g} \quad \text{on } \Gamma, \tag{61}$$

or equivalently,

$$\gamma^0 u \;=\; \mathsf{X}_1\, \gamma^1 u \,+\, \mathsf{X}_1\, \mathbf{g} \quad \text{on } \Gamma. \tag{62}$$

One should notice that $\mathsf{X}_0 \equiv \mathsf{X}_1$ and so henceforth we drop subindices. On the other hand, if u is solution of (60), it holds (cf. Theorem 2)

$$\gamma^i u \;=\; \mathsf{C}_i\, \gamma^i u \;=\; \left(\frac{1}{2}\,\mathsf{Id} + \mathsf{A}_i\right) \gamma^i u \quad \Longrightarrow \quad \frac{1}{2}\gamma^i u \;=\; \mathsf{A}_i\, \gamma^i u\,, \quad i = 0, 1. \tag{63}$$

We now depart from the standard “thought track” and combine (63) with (61) and (62) to obtain

$$\gamma^0 u \,-\, \mathsf{X}\,\gamma^1 u \;=\; 2\,\mathsf{A}_0\,\gamma^0 u \,-\, \mathsf{X}\,\gamma^1 u \;=\; \mathsf{X}\,\mathbf{g}, \tag{64}$$

$$-\mathsf{X}\,\gamma^0 u \,+\, \gamma^1 u \;=\; -\mathsf{X}\,\gamma^0 u \,+\, 2\,\mathsf{A}_1\,\gamma^1 u \;=\; -\mathbf{g}. \tag{65}$$

Identifying $\boldsymbol{\lambda}^i = \gamma^i u$, the Helmholtz problem (60) can be cast in the variational form:

Problem 3 (MTF for $N = 1$) Seek $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1) \in \mathbb{V}_1 := \mathbf{V}_0 \times \mathbf{V}_1$ such that:

$$\mathsf{m}_1(\boldsymbol{\lambda}, \varphi) := \langle \mathbf{M}_1 \boldsymbol{\lambda}, \varphi \rangle_\times = \left\langle \frac{1}{2} \begin{pmatrix} \mathsf{X}\,\mathbf{g} \\ -\mathbf{g} \end{pmatrix}, \varphi \right\rangle_\times, \quad \forall \varphi \in \mathbb{V}_1, \tag{66}$$

where $\mathbf{M}_1 := \begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\mathsf{X} \\ -\frac{1}{2}\mathsf{X} & \mathsf{A}_1 \end{pmatrix} : \mathbb{V}_1 \longrightarrow \mathbb{V}_1.$

Remark 8 Henceforth, we will compare both theoretically and numerically the MTF to the single trace formulation (STF). For $N = 1$, if interior traces $\gamma^i u$ are again denoted by $\boldsymbol{\lambda}^i$, one can arbitrarily choose either $\boldsymbol{\lambda}^0$ or $\boldsymbol{\lambda}^1$ as the unknown. Let us state the problem in terms of $\boldsymbol{\lambda}^0$ and eliminate $\boldsymbol{\lambda}^1$ in (64) via the Calderón identity (63) using (61):

$$\begin{aligned} 2\,\mathsf{A}_0\,\boldsymbol{\lambda}^0 - \mathsf{X}\,\boldsymbol{\lambda}^1 &= 2\,\mathsf{A}_0\,\boldsymbol{\lambda}^0 - 2\,\mathsf{X}\,\mathsf{A}_1\,\boldsymbol{\lambda}^1 = \mathsf{X}\,\mathbf{g}, \\ \Longrightarrow \quad \mathsf{A}_0\,\boldsymbol{\lambda}^0 - \mathsf{X}\,\mathsf{A}_1(\mathsf{X}\,\boldsymbol{\lambda}^0 - \mathbf{g}) &= \frac{1}{2}\mathsf{X}\,\mathbf{g}. \end{aligned} \tag{67}$$

which, after rearranging terms, yields

Problem 4 (STF for $N = 1$) Seek $\boldsymbol{\lambda}^0 \in \mathbf{V}_0$ such that:

$$\left\langle (\mathsf{A}_0 - \mathsf{X}\,\mathsf{A}_1\,\mathsf{X})\,\boldsymbol{\lambda}^0, \varphi^0 \right\rangle_\times = \left\langle \mathsf{X}\,\mathsf{C}_1^c\,\mathbf{g}, \varphi^0 \right\rangle_\times, \quad \forall \varphi^0 \in \mathbf{V}_0. \tag{68}$$

where $\boldsymbol{\lambda}^0 = \gamma^0 u$, with u being the solution of the inhomogeneous transmission problem (60).

The STF, although written differently in [49], is proved to be unique and stable, and is equivalent to the above form. Indeed, the operator product $\mathsf{X}\,\mathsf{A}_1\,\mathsf{X}$ preserves the right orientations as expected, and, on the right-hand side, the exterior Calderón projector multiplied by X is also consistently oriented.

3.1.1 Uniqueness of solutions of Problem 3

Theorem 4 *The multiple trace formulation of Problem* 3 *admits at most one solution.*

Proof We need to show that for $\mathbf{g} = \mathbf{0}$, from

$$\begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\mathsf{X} \\ -\frac{1}{2}\mathsf{X} & \mathsf{A}_1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}^0 \\ \boldsymbol{\lambda}^1 \end{pmatrix} = \mathbf{0} \tag{69}$$

we conclude $\boldsymbol{\lambda}^1 = \boldsymbol{\lambda}^0 = 0$. Let $\boldsymbol{\lambda}^i$ be trace solutions of (69). By Theorem 1, we can define the (radiating) Helmholtz solutions

$$u_1 := \Psi^1(\boldsymbol{\lambda}^1) \quad \text{on } \Omega_1 \quad \text{and} \quad u_0 := \Psi^0(\boldsymbol{\lambda}^0) \quad \text{on } \Omega_0 . \tag{70}$$

Taking interior traces yields

$$\gamma^0 u_0 = \left(\frac{1}{2}\,\mathsf{Id} + \mathsf{A}_0\right)\boldsymbol{\lambda}^0 = \frac{1}{2}\left(\boldsymbol{\lambda}^0 + \mathsf{X}\,\boldsymbol{\lambda}^1\right), \tag{71}$$

$$\gamma^1 u_1 = \left(\frac{1}{2}\,\mathsf{Id} + \mathsf{A}_1\right)\boldsymbol{\lambda}^1 = \frac{1}{2}\left(\boldsymbol{\lambda}^1 + \mathsf{X}\,\boldsymbol{\lambda}^0\right), \tag{72}$$

where the second equalities are due to (69). Hence, the trace jump

$$[\gamma u]_\Gamma = \frac{1}{2}\mathsf{X}\left(\boldsymbol{\lambda}^0 + \mathsf{X}\,\boldsymbol{\lambda}^1\right) - \frac{1}{2}\left(\boldsymbol{\lambda}^1 + \mathsf{X}\,\boldsymbol{\lambda}^0\right) = \mathbf{0} \tag{73}$$

since $\mathsf{X}^2 = \mathsf{Id}$. Consequently, we conclude that

$$u := \begin{cases} u_0, & \text{in } \Omega_0, \\ u_1, & \text{in } \Omega_1, \end{cases} \tag{74}$$

is a Helmholtz solution over the whole $\mathbb{R}^d$. By uniqueness of the Helmholtz radiating solution [26, 28, 37], [49, Section 2], it holds $u_0 \equiv 0$ and $u_1 \equiv 0$ so that

$$\boldsymbol{\lambda}^1 + \mathsf{X}\,\boldsymbol{\lambda}^0 = 0 \tag{75}$$

and

$$\mathsf{C}_0\,\boldsymbol{\lambda}^0 = 0 \quad \text{and} \quad \mathsf{C}_1\,\boldsymbol{\lambda}^1 = 0. \tag{76}$$

Thus, $\boldsymbol{\lambda}^0$ is Cauchy data in Ω_0^c for a wavenumber κ_0, i.e. $\boldsymbol{\lambda}^0 \in \operatorname{Ran}\mathsf{C}_0^c$. Similarly, $\boldsymbol{\lambda}_1$ is also Cauchy data in Ω_1^c for κ_1. Hence, let us now construct the following radiating Helmholtz solutions:

$$u_0^c := \Psi^0(\boldsymbol{\lambda}^0) \quad \text{on } \Omega_0^c \equiv \Omega_1, \tag{77}$$

$$u_1^c := -\Psi^1(\boldsymbol{\lambda}^1) \quad \text{on } \Omega_1^c \equiv \Omega_0, \tag{78}$$

and define

$$u^c := \begin{cases} u_1^c, & \text{in } \Omega_0, \\ u_0^c, & \text{in } \Omega_1. \end{cases} \tag{79}$$

We now take interior traces over Γ:

$$\gamma^1 u_0^c = \mathsf{X}\,\gamma^{0,c} u_0^c = \mathsf{X}\left(\frac{1}{2}\,\mathsf{Id} - \mathsf{A}_0\right)\boldsymbol{\lambda}^0 = \frac{1}{2}\,\mathsf{X}\left(\boldsymbol{\lambda}^0 - \mathsf{X}\,\boldsymbol{\lambda}^1\right) = \frac{1}{2}\left(\mathsf{X}\,\boldsymbol{\lambda}^0 - \boldsymbol{\lambda}^1\right),$$

$$\gamma^0 u_1^c = \mathsf{X}\,\gamma^{1,c} u_1^c = -\mathsf{X}\left(\frac{1}{2}\,\mathsf{Id} - \mathsf{A}_1\right)\boldsymbol{\lambda}^1 = -\frac{1}{2}\,\mathsf{X}\left(\boldsymbol{\lambda}^1 - \mathsf{X}\,\boldsymbol{\lambda}^0\right) = \frac{1}{2}\left(-\mathsf{X}\,\boldsymbol{\lambda}^1 + \boldsymbol{\lambda}^0\right),$$

and again it holds $\left[\gamma u^c\right]_\Gamma = 0$ implying that u^c is also a radiating solution of a Helmholtz problem with discontinuous coefficients over the entire space. By analytic continuation [26, 28, 37], [49, Section 2], uniqueness is guaranteed and the entire solution is equal to zero and, in particular, its trace vanishes

$$-\boldsymbol{\lambda}^1 + \mathsf{X}\,\boldsymbol{\lambda}^0 \;=\; 0\,, \tag{80}$$

which, together with (75), yields $\boldsymbol{\lambda}_1 = 0$ and $\boldsymbol{\lambda}_0 = 0$. □

3.1.2 Coercivity of $\mathbf{M}_1$ *and stability*

Theorem 5 *The operator* $\mathbf{Q}\,\mathbf{M}_1 : \mathbb{V}_1 \longrightarrow \mathbf{V}_0' \times \mathbf{V}_1'$ *is coercive, where*

$$\mathbf{Q} \;:=\; \begin{pmatrix} \mathsf{Q}_0 & 0 \\ 0 & \mathsf{Q}_1 \end{pmatrix} \;:\; \mathbb{V}_1 \longrightarrow \mathbf{V}_0' \times \mathbf{V}_1', \tag{81}$$

i.e. for all $\boldsymbol{\lambda} \in \mathbb{V}_1$ *there exists a constant* $\alpha_{\mathbf{M}_1}$ *such that*

$$\mathfrak{Re}\left\{\left(\boldsymbol{\lambda}\,,\;\left(\mathbf{M}_1 + \mathbf{T}_{\mathbf{M}_1}\right)\boldsymbol{\lambda}\right)_\times\right\} \;\geq\; \alpha_{\mathbf{M}_1}\,\|\boldsymbol{\lambda}\|^2_{\mathbb{V}_1}\,, \quad \forall\,\boldsymbol{\lambda} \in \mathbb{V}_1. \tag{82}$$

where $\mathbf{T}_{\mathbf{M}_1} : \mathbb{V}_1 \rightarrow \mathbb{V}_1$ *is compact.*

Proof Let us study the sesquilinear form:

$$\begin{aligned}\left(\begin{pmatrix}\varphi^0\\ \varphi^1\end{pmatrix}, \begin{pmatrix}\mathsf{A}_0 & -\frac{1}{2}\mathsf{X}\\ -\frac{1}{2}\mathsf{X} & \mathsf{A}_1\end{pmatrix}\begin{pmatrix}\varphi^0\\ \varphi^1\end{pmatrix}\right)_\times &= \left(\varphi^0\,,\;\mathsf{A}_0\,\varphi^0\right)_\times - \frac{1}{2}\left(\varphi^0\,,\;\mathsf{X}\,\varphi^1\right)_\times \\ &\quad - \frac{1}{2}\left(\varphi^1\,,\;\mathsf{X}\,\varphi^0\right)_\times + \left(\varphi^1\,,\;\mathsf{A}_1\,\varphi^1\right)_\times,\end{aligned} \tag{83}$$

where the sesquilinear product $(\cdot\,,\,\cdot)_\times$ is defined in Section 2.2 here without the subscript. Cross terms yield only purely imaginary terms as

$$\left(\varphi^0\,,\;\mathsf{X}\,\varphi^1\right)_\times \;=\; \left(\mathsf{X}^\dagger\,\varphi^0\,,\;\varphi^1\right) \;=\; -\left(\mathsf{X}\,\varphi^0\,,\;\varphi^1\right) \;=\; -\overline{\left(\varphi^1\,,\;\mathsf{X}\,\varphi^0\right)}_\times \tag{84}$$

by Lemma 1. Coercivity follows by Theorem 3 applied on the real part of the remaining terms in (83) and defining $\mathbf{T}_{\mathbf{M}_1} := \begin{pmatrix}\mathbf{T}_{\mathsf{A}_0} & 0\\ 0 & \mathbf{T}_{\mathsf{A}_1}\end{pmatrix}$. □

Using the Fredholm alternative together with Theorems 4 and 5 we immediately deduce

Corollary 1 *For all* $\mathbf{g} \in \mathbf{V}_0$*, there exists a unique solution* $\boldsymbol{\lambda} \in \mathbb{V}_1$ *of Problem* 3 *satisfying the stability estimate*

$$\|\boldsymbol{\lambda}\|_{\mathbb{V}_1} \;\leq\; c \left\| \begin{pmatrix} \mathsf{X}\,\mathbf{g} \\ -\mathbf{g} \end{pmatrix} \right\|_{\mathbb{V}_1} \;\leq\; C\,\|\mathbf{g}\|_{\mathbf{V}_0}\,. \tag{85}$$

3.1.3 Mapping properties of $\mathbf{M}_1^{-1}$

Lastly, we study the mapping properties of the MTF solution operator for any excitation term. In particular, for more regular right-hand sides, the solution is expected to become more regular.

Theorem 6 *Recall the definition of the trace space* $\mathbf{V}_i^{1/2}$ (14). *The operator* $\mathbf{T}_{\mathbf{M}_1}$ *is regularizing, i.e.*

$$\mathbf{T}_{\mathbf{M}_1} : \mathbb{V}_1 \longrightarrow \mathbf{V}_0^{1/2} \times \mathbf{V}_1^{1/2}. \tag{86}$$

Proof This is an immediate consequence of Lemma 6 and the definition of $\mathbf{T}_{\mathbf{M}_1}$ (cf. Proof of Theorem 5). □

Theorem 7 *Let* $\mathbf{g}_i \in \mathbf{V}_i^{1/2}$ *for* $i = 0, 1$ *be boundary data. Then, the solution components of the MTF,* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1)$*, satisfying*

$$\mathsf{m}_1(\boldsymbol{\lambda}, \varphi) \;=\; \left\langle \begin{pmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \end{pmatrix}, \varphi \right\rangle_{\times}, \quad \forall\, \varphi \in \mathbb{V}_1, \tag{87}$$

also lie in $\mathbf{V}_0^{1/2} \times \mathbf{V}_1^{1/2}$.

Proof The proof is based on the regularity results for the STF. Therefore, a connection must be made between both formulations which is immediate when the right-hand side has the structure presented in the formulation of Problem 3 as shown Remark 8. However, for arbitrary sources, as in (87), we must rearrange terms adequately. Let us try to derive the associated STF formulation from (87). For this, we proceed as follows

$$\begin{aligned} \begin{pmatrix} \mathsf{X} & \mathbf{0} \\ \mathbf{0} & 2\,\mathsf{A}_1 \end{pmatrix} \begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\,\mathsf{X} \\ -\frac{1}{2}\,\mathsf{X} & \mathsf{A}_1 \end{pmatrix} \begin{pmatrix} \lambda^0 \\ \lambda^1 \end{pmatrix} &= \begin{pmatrix} \mathsf{X} & \mathbf{0} \\ \mathbf{0} & 2\,\mathsf{A}_1 \end{pmatrix} \begin{pmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \end{pmatrix}, \\ \begin{pmatrix} \mathsf{X}\,\mathsf{A}_0 & -\frac{1}{2}\,\mathsf{Id} \\ -\,\mathsf{A}_1\,\mathsf{X} & \frac{1}{2}\,\mathsf{Id} \end{pmatrix} \begin{pmatrix} \lambda^0 \\ \lambda^1 \end{pmatrix} &= \begin{pmatrix} \mathsf{X}\,\mathbf{g}_0 \\ 2\,\mathsf{A}_1\,\mathbf{g}_1 \end{pmatrix}, \end{aligned} \tag{88}$$

where in the last step, we have used identity (57). Adding both equations yields

$$(\mathsf{X}\,\mathsf{A}_0 - \mathsf{A}_1\,\mathsf{X})\,\boldsymbol{\lambda}_0 \;=\; \mathsf{X}\,\mathbf{g}_0 + 2\,\mathsf{A}_1\,\mathbf{g}_1, \tag{89}$$

or equivalently,

$$(\mathsf{A}_0 - \mathsf{X}\,\mathsf{A}_1\,\mathsf{X})\,\boldsymbol{\lambda}_0 \;=\; \mathbf{g}_0 + 2\,\mathsf{X}\,\mathsf{A}_1\,\mathbf{g}_1. \tag{90}$$

This last equation corresponds to the STF for $\boldsymbol{\lambda}_0$ (cf. Problem 4) with well-defined jump data. If $\partial\Omega$ is smooth, standard regularity results for pseudo-differential operators ensure a solution $\boldsymbol{\lambda}_0 \in \mathbf{V}_0^{1/2}$ for our choice of $(\mathbf{g}_0, \mathbf{g}_1)$ [8, Section 6]. In the same reference, Theorem 6.1 allows to conclude the stated result for two-dimensional Lipschitz boundaries since Dirichlet and Neumann jumps are independent on material constants. Similar results hold for 3-D piecewise smooth boundaries (see Remark 9). □

Remark 9 There is a vast literature on local and global regularity results for elliptic transmission problems, see [9, 24, 25, 27, 38, 40] to cite a few. The main observation is that volume solutions can be decomposed into a regular term plus a series of "singular functions" associated to a particular corner or edge. These functions are independent on the data and vanish outside a neighborhood of their respective corner or edges. For the HTP studied, jumps occur only on the zeroth order term, namely in the wavenumbers κ_i, and, consequently, they will not affect the smoothness of solutions much. On the other hand, for Lipschitz polygons or polyhedra there is a Dirichlet trace mapping $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ for $1 \leq s < 2$. Hence, by choosing $s = 3/2$ we retrieve the result of Theorem 7.

3.2 Composite scatterers ($N > 1$)

We now consider the problem of wave propagation in non-homogeneous or composite scatterers modeled by the Helmholtz transmission problem, see (1):

Problem 5 Seek $u \in H^1_{\mathrm{loc}}(\Omega \cup \Omega_0)$ such that

$$\begin{cases} \mathsf{P}_i u := -(\Delta + \kappa_i^2)u \; = \; 0 & \text{in } \Omega_i \quad i = 0, \ldots, N, \\ [\gamma u] \; = \; \mathbf{g} & \text{on } \partial\Omega, \\ [\gamma u] \; = \; \mathbf{0} & \text{on } \Sigma_0, \\ + \text{ radiation conditions} & \text{for } |\mathbf{x}| \longrightarrow \infty, \end{cases} \tag{91}$$

where the boundary data $\mathbf{g} = (g_{\mathrm{D}}, g_{\mathrm{N}}) \in \mathbf{V}_0$ is given.

Extension of the method presented in Section 3.1 is not straightforward due to the presence of "triple points", see Fig. 1. Indeed, in order to write down transmission conditions per subdomain interface Γ_{ij} requires the ability to extend by zero and restrict trace functions defined over the entire subdomain boundary $\partial\Omega_i$. Therefore we have to resort to the piecewise spaces $\widetilde{H}^{\pm 1/2}_{\mathrm{pw}}(\partial\Omega_i)$ introduced in Section 2.2.2.

3.2.1 Functional space setting

Recall that the normal vector $\mathbf{n}^i$ on each $\partial\Omega_i$ points to the exterior of Ω_i. The space for total Cauchy data is

$$\mathbb{V}_N := \mathbf{V}_0 \times \cdots \times \mathbf{V}_N \tag{92}$$

with $\mathbf{V}_i$ defined in Section 2.2. Hence, if $\mathbf{Q}_N := \operatorname{diag}\{\mathsf{Q}_i\}_{i=0}^N$, the dual space of $\mathbb{V}_N$ is $\mathbb{V}'_N := \mathbf{Q}_N \mathbb{V}_N$ with duality pairing given naturally as follows. Let $\varphi = (\varphi^0, \ldots, \varphi^N)$ and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^0, \ldots, \boldsymbol{\lambda}^N)$ belong to $\mathbb{V}_N$, then we write

$$\langle \boldsymbol{\lambda}\,,\, \varphi \rangle_\times := \sum_{i=0}^{N} \left\langle \boldsymbol{\lambda}^i\,,\, \varphi^i \right\rangle_{\times,i} \tag{93}$$

to which we also associate a sesquilinear form $(\,,\,)_\times$ based on individual forms $(\,,\,)_{\times,i}$. We call φ^i the "component"-projection onto $\mathbf{V}_i$ of φ. Equivalently, we define the product spaces

$$\mathbb{V}_{\mathrm{pw},N} := \mathbf{V}_{\mathrm{pw},0} \times \ldots \times \mathbf{V}_{\mathrm{pw},N}, \quad \widetilde{\mathbb{V}}_N := \widetilde{\mathbf{V}}_0 \times \ldots \times \widetilde{\mathbf{V}}_N, \quad \approx{\mathbb{V}}_N := \approx{\mathbf{V}}_0 \times \ldots \times \approx{\mathbf{V}}_N,$$

satisfying $\overset{\approx}{\mathbb{V}}_N \subset \widetilde{\mathbb{V}}_N \subset \mathbb{V}_N \subset \mathbb{V}_{\mathrm{pw},N}$ with $\mathbf{V}_{\mathrm{pw},i}$, $\widetilde{\mathbf{V}}_i$, $\overset{\approx}{\mathbf{V}}_i$ defined in (23)–(25), respectively.

3.2.2 Restriction and extension operators

Transmission conditions are stated in weak sense by restricting test functions $\varphi \in \overset{\approx}{\mathbb{V}}_N$ over interfaces Γ_{ij}. For this, we start by introducing the following operators

$$\text{restriction:} \quad \mathsf{R}_{ij}^{\mathrm{D}} \;:\; H^{1/2}_{\mathrm{pw}}(\partial\Omega_i) \longrightarrow H^{1/2}(\Gamma_{ij}), \tag{94}$$

$$\text{extension by zero:} \quad \mathsf{E}_{ij}^{\mathrm{D}} \;:\; H^{1/2}(\Gamma_{ij}) \longrightarrow H^{1/2}_{\mathrm{pw}}(\partial\Omega_i), \tag{95}$$

satisfying

$$\left\langle \lambda\,,\, \mathsf{R}_{ij}^{\mathrm{D}}\, \varphi \right\rangle_{ij} = \langle \lambda\,,\, \varphi \rangle_{ij}\,, \quad \forall\, \lambda \in \widetilde{H}^{-1/2}(\Gamma_{ij}),\ \varphi \in H^{1/2}_{\mathrm{pw}}(\partial\Omega_i), \tag{96}$$

$$\left\langle \lambda\,,\, \mathsf{E}_{ij}^{\mathrm{D}}\, \varphi \right\rangle_{i} = \langle \lambda\,,\, \varphi \rangle_{ij}\,, \quad \forall\, \lambda \in \widetilde{H}^{-1/2}_{\mathrm{pw}}(\partial\Omega_i),\ \varphi \in H^{1/2}(\Gamma_{ij}), \tag{97}$$

so that

$$\left\langle \lambda\,,\, \mathsf{E}_{ij}^{\mathrm{D}}\, \mathsf{R}_{ij}^{\mathrm{D}}\, \varphi \right\rangle_{i} = \langle \lambda\,,\, \varphi \rangle_{ij}\,, \quad \forall\, \lambda \in \widetilde{H}^{-1/2}_{\mathrm{pw}}(\partial\Omega_i),\ \varphi \in H^{1/2}_{\mathrm{pw}}(\partial\Omega_i). \tag{98}$$

Their dual adjoints are denoted

$$\mathsf{E}_{ij}^{\mathrm{N}} := \left(\mathsf{R}_{ij}^{\mathrm{D}}\right)' : \widetilde{H}^{-1/2}(\Gamma_{ij}) \longrightarrow \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i), \tag{99}$$

$$\mathsf{R}_{ij}^{\mathrm{N}} := \left(\mathsf{E}_{ij}^{\mathrm{D}}\right)' : \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i) \longrightarrow \widetilde{H}^{-1/2}(\Gamma_{ij}), \tag{100}$$

where $\mathsf{E}_{ij}^{\mathrm{N}}$ can be interpreted as the extension by zero for Neumann data over $\partial\Omega_i$ since by definition

$$\left\langle \mathsf{E}_{ij}^{\mathrm{N}}\,\psi\,,\,\varphi\right\rangle_i = \left\langle \left(\mathsf{R}_{ij}^{\mathrm{D}}\right)'\psi\,,\,\varphi\right\rangle_i = \left\langle \psi\,,\,\mathsf{R}_{ij}^{\mathrm{D}}\,\varphi\right\rangle_{ij},$$

for all $\psi \in \widetilde{H}^{-1/2}(\Gamma_{ij})$, $\varphi \in H_{\mathrm{pw}}^{1/2}(\partial\Omega_i)$. Similarly, $\mathsf{R}_{ij}^{\mathrm{N}}$ can be understood as the restriction operator for elements in $\widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i)$. Let now R_{ij} be the linear restriction operator over Γ_{ij} mapping $H_{\mathrm{pw}}^{1/2}(\partial\Omega_i) \times \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i) \to \widetilde{\mathbf{V}}_{ij}$ and defined as follows, cf. (13)

$$\mathsf{R}_{ij}\,\varphi^i := \begin{cases} \begin{pmatrix} \mathsf{R}_{ij}^{\mathrm{D}} & 0 \\ 0 & \mathsf{R}_{ij}^{\mathrm{N}} \end{pmatrix}\varphi^i & \text{if } j \in \Lambda_i, \\ \mathbf{0} & \text{any other case,} \end{cases} \tag{101}$$

for all $j \in \{0, \ldots, N\}$. It also satisfies the obvious mapping property $\mathbf{V}_{\mathrm{pw},i} \to \mathbf{V}_{ij}$. The adjoint operator $\mathsf{R}_{ij}' : \widetilde{\mathbf{V}}_{ij}' \to \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i) \times H_{\mathrm{pw}}^{1/2}(\partial\Omega_i)$ is the formal extension by zero, since by definition, if $j \in \Lambda_i$, for all $\varphi \in \widetilde{\mathbf{V}}_{ij}'$ it holds

$$\begin{aligned} \left\langle \mathsf{R}_{ij}'\,\varphi\,,\,\boldsymbol{\lambda}\right\rangle_{\times,i} &= \left\langle \left(\mathsf{R}_{ij}^{\mathrm{D}}\right)'\varphi_{\mathrm{N}}\,,\,\lambda_{\mathrm{D}}\right\rangle_i + \left\langle \left(\mathsf{R}_{ij}^{\mathrm{N}}\right)'\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}}\right\rangle_i \\ &= \left\langle \mathsf{E}_{ij}^{\mathrm{N}}\,\varphi_{\mathrm{N}}\,,\,\lambda_{\mathrm{D}}\right\rangle_i + \left\langle \mathsf{E}_{ij}^{\mathrm{D}}\,\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}}\right\rangle_i \end{aligned} \tag{102}$$

or, succinctly,

$$\mathsf{R}_{ij}'\,\varphi^i := \begin{cases} \begin{pmatrix} \mathsf{E}_{ij}^{\mathrm{N}} & 0 \\ 0 & \mathsf{E}_{ij}^{\mathrm{D}} \end{pmatrix}\varphi^i & \text{if } j \in \Lambda_i, \\ \mathbf{0} & \text{any other case,} \end{cases} \tag{103}$$

for all $j \in \{0, \ldots, N\}$. Now, the $\times$-adjoint given by

$$\mathsf{R}_{ij}^{\dagger} = \mathsf{Q}_i\,\mathsf{R}_{ij}'\,\mathsf{Q}_j : \widetilde{\mathbf{V}}_{ij} \longrightarrow H_{\mathrm{pw}}^{1/2}(\partial\Omega_i) \times \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i) \tag{104}$$

should be interpreted in weak sense, i.e. if $\varphi \in \widetilde{\mathbf{V}}_{ij}$ and $\boldsymbol{\lambda} \in \widetilde{\mathbf{V}}_i$ one has

$$\begin{aligned} \left(\mathsf{R}_{ij}^{\dagger}\,\varphi\,,\,\boldsymbol{\lambda}\right)_{\times,i} &= \left(\mathsf{E}_{ij}^{\mathrm{D}}\,\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}}\right)_i + \left(\mathsf{E}_{ij}^{\mathrm{N}}\,\varphi_{\mathrm{N}}\,,\,\lambda_{\mathrm{D}}\right)_i \\ &= (\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}})_{ij} + \left(\varphi_{\mathrm{N}}\,,\,\mathsf{R}_{ij}^{\mathrm{D}}\,\lambda_{\mathrm{D}}\right)_{ij} = \left(\varphi\,,\,\mathsf{R}_{ij}\,\boldsymbol{\lambda}\right)_{\times,ij} \end{aligned} \tag{105}$$

which is well defined for the dual pairings $H^{1/2}(\Gamma_{ij}) \times \widetilde{H}^{-1/2}(\Gamma_{ij})$ with sesquilinear form $(\cdot\,,\,\cdot)_{\times,ij}$. Similarly, it holds $\mathsf{R}_{ij}^{\dagger} : \mathbf{V}_{ij} \to \mathbf{V}_{\mathrm{pw},i}$ now testing with dual space functions, i.e. functions in $\widetilde{\widetilde{\mathbf{V}}}_i$. With it, we have the following result:

Lemma 7 *The identity*

$$\sum_{j=0}^{N} \mathsf{R}_{ij}^{\dagger}\,\mathsf{R}_{ij} = \mathsf{Id}_i \tag{106}$$

holds weakly over $\mathbf{V}_i$ *and* $\mathbf{V}_{pw,i}$*, when testing with functions in* $\widetilde{\widetilde{\mathbf{V}}}_i$*. It also holds over* $\widetilde{\mathbf{V}}_i$ *if tested against* $\widetilde{\mathbf{V}}_i$*-functions.*

Proof Let $\varphi, \boldsymbol{\lambda} \in \widetilde{\mathbf{V}}_i$. Then, by property (98) and (105), we observe that

$$\begin{aligned}\left(\mathsf{R}_{ij}^{\dagger}\,\mathsf{R}_{ij}\,\varphi\,,\,\boldsymbol{\lambda}\right)_{\times,i} &= \left(\mathsf{E}_{ij}^{\mathrm{D}}\,\mathsf{R}_{ij}^{\mathrm{D}}\,\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}}\right)_i + \left(\mathsf{E}_{ij}^{\mathrm{N}}\,\mathsf{R}_{ij}^{\mathrm{N}}\,\varphi_{\mathrm{N}}\,,\,\lambda_{\mathrm{D}}\right)_i \\ &= (\varphi_{\mathrm{D}}\,,\,\lambda_{\mathrm{N}})_{ij} + \left(\varphi_{\mathrm{N}}\,,\,\mathsf{E}_{ij}^{\mathrm{D}}\,\mathsf{R}_{ij}^{\mathrm{D}}\,\lambda_{\mathrm{D}}\right)_i \;=\; (\varphi\,,\,\boldsymbol{\lambda})_{\times,ij}\end{aligned} \tag{107}$$

or equal to zero if $j \notin \Lambda_i$. Hence,

$$\sum_{j=0}^{N} \left(\mathsf{R}_{ij}^{\dagger}\,\mathsf{R}_{ij}\,\varphi\,,\,\boldsymbol{\lambda}\right)_{\times,i} \;=\; \sum_{j\in\Lambda_i} (\varphi\,,\,\boldsymbol{\lambda})_{\times,ij} = \;(\varphi\,,\,\boldsymbol{\lambda})_{\times,i} \tag{108}$$

as stated. For $\varphi \in \mathbf{V}_{\mathrm{pw},i}$ or in $\mathbf{V}_i$ and $\boldsymbol{\lambda} \in \widetilde{\widetilde{\mathbf{V}}}_i$ the proof is exactly the same. □

3.2.3 Multiple traces formulation (MTF)

Expansion of the jump operator in (91) over each interface Γ_{ij} in Σ in weak sense yields

$$\left(\mathsf{R}_{0i}\,\gamma^0 u - \mathsf{R}_{i0}\,\mathsf{X}_i\,\gamma^i u\,,\,\mathsf{R}_{0i}\,\varphi^0\right)_{\times,0i} \;=\; \left(\mathsf{R}_{0i}\,\mathsf{X}_0\,\mathbf{g}\,,\,\mathsf{R}_{0i}\,\varphi^0\right)_{\times,0i}, \tag{109a}$$

$$\left(-\,\mathsf{R}_{0j}\,\mathsf{X}_0\,\gamma^0 u + \mathsf{R}_{j0}\,\gamma^j u\,,\,\mathsf{R}_{j0}\,\varphi^j\right)_{\times,j0} \;=\; -\left(\mathsf{R}_{0j}\,\mathbf{g}\,,\,\mathsf{R}_{j0}\,\varphi^j\right)_{\times,j0}, \tag{109b}$$

$$\left(\mathsf{R}_{ji}\,\gamma^j u - \mathsf{R}_{ij}\,\mathsf{X}_i\,\gamma^i u\,,\,\mathsf{R}_{ji}\,\varphi^j\right)_{\times,ji} \;=\; 0, \tag{109c}$$

$1 \leq i \neq j \leq N$, for all $\varphi \in \widetilde{\widetilde{\mathbb{V}}}_N$ and where restrictions operators have been used accordingly, cf. (61) and (62) for the case $N = 1$. Notice that we have used

$\Gamma_{ij} = \Gamma_{ji}$ and doubled the number of transmission conditions. Moreover, since $\gamma^i u \in \mathbf{V}_i$ their restriction $\mathsf{R}_{ij}\,\gamma^i u \in \mathbf{V}_{ij}$ while the restrictions $\mathsf{R}_{ji}\,\varphi^j \in \widetilde{\widetilde{\mathbf{V}}}_{ij}$. Thus, we can extend conditions (109) over each $\partial\Omega_i$ in order to define transmission conditions as follows:

$$\left(\gamma^0 u\,,\, \mathsf{R}^\dagger_{0i}\,\mathsf{R}_{0i}\,\varphi^0\right)_{\times,0} - \left(\mathsf{R}^\dagger_{0i}\,\mathsf{R}_{i0}\,\mathsf{X}_i\,\gamma^i u\,,\, \varphi^0\right)_{\times,0} = \left(\mathsf{X}_0\,\mathbf{g}\,,\, \mathsf{R}^\dagger_{0i}\,\mathsf{R}_{0i}\,\varphi^0\right)_{\times,0}, \tag{110a}$$

$$-\left(\mathsf{R}^\dagger_{j0}\,\mathsf{R}_{0j}\,\mathsf{X}_0\,\gamma^0 u\,,\, \varphi^j\right)_{\times,j} + \left(\gamma^j u\,,\, \mathsf{R}^\dagger_{j0}\,\mathsf{R}_{j0}\,\varphi^j\right)_{\times,j} = -\left(\mathsf{R}^\dagger_{j0}\,\mathsf{R}_{0j}\,\mathbf{g}\,,\, \varphi^j\right)_{\times,j}, \tag{110b}$$

$$\left(\gamma^j u\,,\, \mathsf{R}^\dagger_{ji}\,\mathsf{R}_{ji}\,\varphi^j\right)_{\times,j} - \left(\mathsf{R}^\dagger_{ji}\,\mathsf{R}_{ij}\,\mathsf{X}_i\,\gamma^i u\,,\, \varphi^j\right)_{\times,j} = 0, \tag{110c}$$

$1 \leq i \neq j \leq N$, for all $\varphi \in \widetilde{\widetilde{\mathbb{V}}}_N$. For simplicity, we introduce the operator

$$\widetilde{\mathsf{X}}_{ji} := \mathsf{R}^\dagger_{ji}\,\mathsf{R}_{ij}\,\mathsf{X}_i \;:\; \mathbf{V}_i \longrightarrow \mathbf{V}_{\mathrm{pw},j} \quad (\text{or} \quad \widetilde{\mathbf{V}}_i \longrightarrow \widetilde{\mathbf{V}}_j). \tag{111}$$

Again, it should be interpreted in weak sense. We use Lemma 7 to simplify the sum of conditions (110a) with respect to i to obtain:

$$\left(\gamma^0 u\,,\, \varphi^0\right)_{\times,0} - \sum_{i=1}^{N} \left(\widetilde{\mathsf{X}}_{0i}\gamma^i u\,,\, \varphi^0\right)_{\times,0} = \left(\mathsf{X}_0\,\mathbf{g}\,,\, \varphi^0\right)_{\times,0}, \quad \forall\, \varphi^0 \in \widetilde{\widetilde{\mathbf{V}}}_0, \tag{112}$$

while the sum $\sum_i$(110c)+(110b) with respect to i yields, for all $\varphi^j \in \widetilde{\widetilde{\mathbf{V}}}_j$,

$$\left(\gamma^j u\,,\, \varphi^j\right)_{\times,j} - \sum_{\substack{i=0\\ i\neq j}}^{N} \left(\widetilde{\mathsf{X}}_{ji}\gamma^i u\,,\, \varphi^j\right)_{\times,j} = -\left(\mathsf{R}^\dagger_{j0}\,\mathsf{R}_{0j}\,\mathbf{g}\,,\, \varphi^j\right)_{\times,j}, \tag{113}$$

for fixed $j \in \{1, \ldots, N\}$. These last steps are not necessary in the case of a single homogeneous scatterer (Section 3.1) since there is a single interface. On the other hand, by Theorem 2 solutions of Problem 5 must satisfy

$$\gamma^i u \;=\; \mathsf{C}_i\,\gamma^i u \;=\; \left(\frac{1}{2}\,\mathsf{Id} + \mathsf{A}_i\right)\gamma^i u \quad\Longrightarrow\quad \frac{1}{2}\gamma^i u \;=\; \mathsf{A}_i\,\gamma^i u \quad \text{on } \partial\Omega_i, \tag{114}$$

$i = 0, \ldots, N$, due to the properties of Calderón projectors (see Section 2.3.3). Replacing the above in (112) and (113), and identifying $\boldsymbol{\lambda}^i = \gamma^i u$, gives a boundary integral formulation for Problem 5, which extends Problem 3 to $N > 1$:

Problem 6 (MTF) Seek $\boldsymbol{\lambda} \in \mathbb{V}_N$ such that the variational form:

$$\mathsf{m}_N(\boldsymbol{\lambda}, \varphi) := (\mathbf{M}_N \boldsymbol{\lambda}, \varphi)_\times = \frac{1}{2} \left(\begin{pmatrix} \mathsf{X}_0 \mathbf{g} \\ -\mathsf{R}_{10}^\dagger \mathsf{R}_{01} \mathbf{g} \\ \vdots \\ -\mathsf{R}_{N0}^\dagger \mathsf{R}_{0N} \mathbf{g} \end{pmatrix}, \varphi \right)_\times , \quad \text{forall } \varphi \in \widetilde{\widetilde{\mathbb{V}}}_N, \tag{115}$$

is satisfied for $\mathbf{g} \in \mathbf{V}_0$ with

$$\mathbf{M}_N := \begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\widetilde{\mathsf{X}}_{01} & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{0N} \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{10} & \mathsf{A}_1 & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{N0} & -\frac{1}{2}\widetilde{\mathsf{X}}_{N1} & \cdots & \mathsf{A}_N \end{pmatrix} : \mathbb{V}_N \longrightarrow \mathbb{V}_{\mathrm{pw},N}. \tag{116}$$

Remark 10 Now it becomes clear why we have to resort to the spaces $\widetilde{\widetilde{\mathbf{V}}}$ instead of $\mathbf{V}$ for test functions. If both $\boldsymbol{\lambda}^i$ and $\boldsymbol{\lambda}^j$ belong to $\mathbf{V}_i$ and $\mathbf{V}_j$, respectively, then extension by zero is meaningless and we cannot proceed as in Section 3.1. Nonetheless, observe that for a single scatterer, $N = 1$, we retrieve Problem 3.

3.2.4 Continuity properties

Theorem 8 (Continuity of the MTF) *The sesquilinear form* m_N *from* (115) *is continuous as a mapping* $\mathbb{V}_N \times \widetilde{\widetilde{\mathbb{V}}}_N \to \mathbb{C}$ *as well as on* $\widetilde{\mathbb{V}}_N \times \widetilde{\mathbb{V}}_N$.

Proof Write the variational form as a sum:

$$\mathsf{m}_N(\boldsymbol{\lambda}, \varphi) = \sum_{i=0}^{N} \left(\mathsf{A}_i \boldsymbol{\lambda}^i, \varphi^i\right)_{\times,i} - \frac{1}{2} \sum_{i=0}^{N} \sum_{\substack{j=0 \\ i \neq j}}^{N} \left(\widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j, \varphi^i\right)_{\times,i}$$

$$|\mathsf{m}_N(\boldsymbol{\lambda}, \varphi)| \leq \sum_{i=0}^{N} \left|\left(\mathsf{A}_i \boldsymbol{\lambda}^i, \varphi^i\right)_{\times,i}\right| + \frac{1}{2} \sum_{i=0}^{N} \sum_{\substack{j=0 \\ i \neq j}}^{N} \left|\left(\widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j, \varphi^i\right)_{\times,i}\right| \tag{117}$$

Each cross term is bounded by continuity of R_{ij}, $\mathsf{R}_{ji}^\dagger$, and X_i as follows

$$\left|\left(\widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j, \varphi^i\right)_{\times,i}\right| = \left(\mathsf{R}_{ji} \mathsf{X}_j \boldsymbol{\lambda}^j, \mathsf{R}_{ij} \varphi^i\right)_{\times,i} \leq C_{ij} \left\|\boldsymbol{\lambda}^j\right\|_{\mathbf{V}_j} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i}, \tag{118}$$

$i \neq j$, while the operator A_i is well known to be continuous from $\mathbf{V}_i$ to itself, and thus the form,

$$\left|\left(\mathsf{A}_i \boldsymbol{\lambda}^i\,,\,\varphi^i\right)_{\times,i}\right| \;\leq\; \left\|\mathsf{A}_i \boldsymbol{\lambda}^i\right\|_{\mathbf{V}_i} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i} \;\leq\; C_{\mathsf{A}_i} \left\|\boldsymbol{\lambda}^i\right\|_{\mathbf{V}_i} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i}. \tag{119}$$

Thus, by rearraging terms

$$\begin{aligned}
|\mathsf{m}_N(\boldsymbol{\lambda},\varphi)| &\leq \sum_{i=0}^{N} C_{\mathsf{A}_i} \left\|\boldsymbol{\lambda}^i\right\|_{\mathbf{V}_i} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i} + \frac{1}{2}\sum_{i=0}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N} C_{ij} \left\|\boldsymbol{\lambda}^j\right\|_{\mathbf{V}_j} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i},\\
&\leq \sum_{i=0}^{N} \left(C_{\mathsf{A}_i} \left\|\boldsymbol{\lambda}^i\right\|_{\mathbf{V}_i} + \frac{1}{2}\sum_{\substack{j=0\\ i\neq j}}^{N} C_{ij} \left\|\boldsymbol{\lambda}^j\right\|_{\mathbf{V}_j}\right) \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i},\\
&\leq \sum_{i=0}^{N}\sum_{j=0}^{N} \tilde{C}_{ij} \left\|\boldsymbol{\lambda}^j\right\|_{\mathbf{V}_j} \left\|\varphi^i\right\|_{\widetilde{\widetilde{\mathbf{V}}}_i} \;\leq\; C^* \left\|\boldsymbol{\lambda}\right\|_{\mathbb{V}_N} \left\|\varphi\right\|_{\widetilde{\widetilde{\mathbb{V}}}_N},
\end{aligned} \tag{120}$$

where $C^* := \max_{i,j}\{\tilde{C}_{ij}\}$ with $\tilde{C}_{ij} = \max\{C_{\mathsf{A}_i}, \frac{1}{2}C_{ij}\}$, which yields the first assertion. The second is established by analoguous considerations. □

3.2.5 Single trace formulation (STF)

The STF can be directly derived from the MTF when restricting test and trial functions to skeleton spaces as introduced in Section 2.2.5. Specifically, for functions $\varphi \in H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$, in weak sense

$$\mathsf{R}_{ij}\,\varphi \;=\; \mathsf{R}_{ji}\,\mathsf{X}_j\,\varphi, \quad 0 \leq i \neq j \leq N. \tag{121}$$

We will denote the *orientation aware* restriction of φ to a subdomain boundary $\partial\Omega_i$ as φ^i. The STF follows by simply testing (115) with φ and using (121). Explicitely,

$$\begin{aligned}
\sum_{i=0}^{N} \left(\mathsf{A}_i \boldsymbol{\lambda}^i\,,\,\varphi^i\right)_{\times,i} - \frac{1}{2}\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N} \left(\widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j\,,\,\varphi^i\right)_{\times,i} \;=\; \tfrac{1}{2}\left(\mathsf{X}_0\,\mathbf{g}\,,\,\varphi^0\right)_{\times,0}&\\
-\tfrac{1}{2}\textstyle\sum_{i=1}^{N}\left(\mathsf{R}_{i0}^{\dagger}\,\mathsf{R}_{0i}\,\mathbf{g}\,,\,\varphi^i\right)_{\times,i}&.
\end{aligned} \tag{122}$$

By (121) the right-hand side becomes

$$\begin{aligned}
\frac{1}{2}\left(\mathsf{X}_0\,\mathbf{g},\varphi^0\right)_{\times,0} - \frac{1}{2}\sum_{i=1}^{N}\left(\mathsf{R}_{0i}\,\mathbf{g}\,,\mathsf{R}_{i0}\,\varphi^i\right)_{\times,i} &= \frac{1}{2}\left(\mathsf{X}_0\,\mathbf{g},\varphi^0\right)_{\times,0} - \frac{1}{2}\sum_{i=1}^{N}\left(\mathsf{R}_{0i}\,\mathbf{g}\,,\mathsf{R}_{0i}\,\mathsf{X}_0\,\varphi^0\right)_{\times,0i}\\
\text{(by Lemma 1)}\quad &= \frac{1}{2}\left(\mathsf{X}_0\,\mathbf{g},\varphi^0\right)_{\times,0} + \frac{1}{2}\sum_{i=1}^{N}\left(\mathsf{X}_0\,\mathsf{R}_{0i}^{\dagger}\,\mathsf{R}_{0i}\,\mathbf{g},\varphi^0\right)_{\times,0}\\
\text{(by Lemma 7)}\quad &= \left(\mathsf{X}_0\,\mathbf{g},\varphi^0\right)_{\times,0}.
\end{aligned}$$

Now, double sum terms in (122) can be written as

$$\begin{aligned}
\left(\widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j, \varphi^i\right)_{\times,i} + \left(\widetilde{\mathsf{X}}_{ji}\boldsymbol{\lambda}^i, \varphi^j\right)_{\times,j} &= \left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j, \mathsf{R}_{ij}\,\varphi\right)_{\times,ij} + \left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i, \mathsf{R}_{ji}\,\varphi\right)_{\times,ij} \\
(121) &= \left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j, \mathsf{R}_{ij}\,\varphi\right)_{\times,ij} + \left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i, \mathsf{R}_{ij}\,\mathsf{X}_i\,\varphi\right)_{\times,ij} \\
(\text{by Lemma 1}) &= \left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j, \mathsf{R}_{ij}\,\varphi\right)_{\times,ij} - \left(\mathsf{R}_{ij}\,\boldsymbol{\lambda}^i, \mathsf{R}_{ij}\,\varphi\right)_{\times,ij} \\
&= \left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j - \mathsf{R}_{ij}\,\boldsymbol{\lambda}^i, \mathsf{R}_{ij}\,\varphi\right)_{\times,ij}.
\end{aligned}$$

Transmission conditions built into $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ for $1 \le i \ne j \le N$, imply that these terms vanish whereas for $i = 0$ and $j = 1, \ldots, N$, we have

$$\left(\mathsf{R}_{j0}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j - \mathsf{R}_{0j}\,\boldsymbol{\lambda}^0, \mathsf{R}_{0j}\,\varphi\right)_{\times,0j} = -\left(\mathsf{R}_{0j}\,\mathsf{X}_0\,\mathbf{g}, \mathsf{R}_{0j}\,\varphi\right)_{\times,0j}, \tag{123}$$

so that

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\left(\widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j, \varphi^i\right)_{\times,i} = -\sum_{j=1}^{N}\left(\mathsf{R}_{0j}\,\mathsf{X}_0\,\mathbf{g}, \mathsf{R}_{0j}\,\varphi\right)_{\times,0j} = -\left(\mathsf{X}_0\,\mathbf{g}, \varphi\right)_{\times,0} \tag{124}$$

again thanks to Lemma 7. Thus, one obtains the following formulation

Problem 7 (STF for $N > 1$) Seek $\boldsymbol{\lambda} \in H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ such that

$$\sum_{i=0}^{N}\left(\mathsf{A}_i\,\boldsymbol{\lambda}^i, \varphi^i\right)_{\times,i} = \frac{1}{2}\left(\mathsf{X}_0\,\mathbf{g}, \varphi^0\right)_{\times,0}, \quad \text{for all } \varphi \in H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma), \tag{125}$$

is satisfied for $\mathbf{g} \in \mathbf{V}_0$.

Finally, notice that for $N = 1$, we recover Problem 4 as

$$\begin{aligned}
\left(\mathsf{A}_0\,\boldsymbol{\lambda}^0, \varphi^0\right)_{\times,0} + \left(\mathsf{A}_1\,\boldsymbol{\lambda}^1, \varphi^1\right)_{\times,1} &= \left(\mathsf{A}_0\,\boldsymbol{\lambda}^0, \varphi^0\right)_{\times,0} + \left(\mathsf{A}_1\,\boldsymbol{\lambda}^1, \mathsf{X}_0\,\varphi^0\right)_{\times,0} \\
&\overset{(121)}{=} \left(\mathsf{A}_0\,\boldsymbol{\lambda}^0 - \mathsf{X}_0\,\mathsf{A}_1\,\boldsymbol{\lambda}^1, \varphi^0\right)_{\times,0} \\
\text{trans. cond.} \quad &= \left(\mathsf{A}_0\,\boldsymbol{\lambda}^0 - \mathsf{X}_0\,\mathsf{A}_1(\mathsf{X}_0\,\boldsymbol{\lambda}^0 - \mathbf{g}), \boldsymbol{\varphi}^0\right)_{\times,0} \\
&= \left((\mathsf{A}_0 - \mathsf{X}_0\,\mathsf{A}_1\,\mathsf{X}_0)\boldsymbol{\lambda}^0, \varphi^0\right)_{\times,0} \\
&\quad + \left(\mathsf{X}_0\,\mathsf{A}_1\,\mathbf{g}, \varphi^0\right)_{\times,0},
\end{aligned}$$

with the term in $\mathbf{g}$ subtracting the r.h.s in (125).

3.2.6 Uniqueness of solutions for Problem 6

Theorem 9 (*c.f.* Theorem 4) *Problem* 6 *admits at most one solution.*

Proof Take $\mathbf{g} = 0$, then one must show that $\boldsymbol{\lambda} \equiv 0$.

(i) The first step of the proof runs parallel to the first part of the proof of Theorem 4. Inside each domain Ω_i use the potential (42) to define (radiating) Helmholtz solutions $u_i := \Psi^i(\boldsymbol{\lambda}^i)$ based on the local components $\boldsymbol{\lambda}^i \in \mathbf{V}_i$ of $\boldsymbol{\lambda} \in \mathbb{V}_N$, for $i = 0, \ldots, N$, and set

$$u(\mathbf{x}) \ := \ u_i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_i \ , \quad i = 0, \ldots, N \ . \tag{126}$$

Take interior traces for u_i on the boundary $\partial\Omega_i$, which, by (47) and the definition (55) yields

$$\gamma^i u_i \ = \ \left(\mathsf{A}_i + \frac{1}{2}\,\mathsf{Id}\right)\boldsymbol{\lambda}^i \overset{(115)}{=} \frac{1}{2}\boldsymbol{\lambda}^i + \frac{1}{2}\sum_{\substack{j=0\\ j\neq i}}^{N} \widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j \ , \tag{127}$$

where we used (115) with $\mathbf{g} = 0$. At an interface Γ_{ij}, we take the dual product between the jump operator and a smooth function φ in $\mathscr{D}(\Gamma_{ij}) \times \mathscr{D}(\Gamma_{ij}) \subset \widetilde{\widetilde{\mathbf{V}}}_{ij}$ compactly supported on Γ_{ij}. Recall (35) to see

$$\left([\gamma u]_{\Gamma_{ij}} \ , \ \varphi\right)_{\times,ij} \ = \ \left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\gamma^j u_j \, , \, \varphi\right)_{\times,ij} - \left(\mathsf{R}_{ij}\,\gamma^i u_i \, , \, \varphi\right)_{\times,ij}. \tag{128}$$

Using (127) and the definition $\widetilde{\mathsf{X}}_{jl} = \mathsf{R}^{\dagger}_{jl}\,\mathsf{R}_{lj}\,\mathsf{X}_l$ from (111), the first term can be expanded into

$$\begin{aligned}
\left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\gamma^j u_j \, , \, \varphi\right)_{\times,ij} &\overset{(127)}{=} \frac{1}{2}\Big(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j + \frac{1}{2}\sum_{\substack{l=0\\ l\neq j}}^{N} \mathsf{R}_{ji}\,\mathsf{X}_j\,\mathsf{R}^{\dagger}_{jl}\,\mathsf{R}_{lj}\,\mathsf{X}_l\,\boldsymbol{\lambda}^l , \varphi\Big)_{\times,ij} \\
&= \frac{1}{2}\left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j \, , \, \varphi\right)_{\times,ij} - \frac{1}{2}\sum_{\substack{l=0\\ l\neq j}}^{N} \underbrace{\left(\mathsf{R}_{lj}\,\mathsf{X}_l\,\boldsymbol{\lambda}^l \, , \, \mathsf{R}_{jl}\,\mathsf{X}_j\,\mathsf{R}^{\dagger}_{ji}\,\varphi\right)_{\times,ij}}_{=0 \text{ for } l\neq i} \\
&= \frac{1}{2}\left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j \, , \, \varphi\right)_{\times,ij} - \frac{1}{2}\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i \, , \, \mathsf{R}_{ji}\,\mathsf{X}_j\,\varphi\right)_{\times,ij} \\
&= \frac{1}{2}\left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j \, , \, \varphi\right)_{\times,ij} + \frac{1}{2}\left(\mathsf{R}_{ij}\,\boldsymbol{\lambda}^i \, , \, \varphi\right)_{\times,ij}
\end{aligned} \tag{129}$$

by Lemma 1 and by definition of R_{ji}. The second term in (128) can be treated similarly and becomes

$$\begin{aligned}
\left(\mathsf{R}_{ij}\,\gamma^i u_i \, , \, \varphi\right)_{\times,ij} \ &= \ \frac{1}{2}\left(\mathsf{R}_{ij}\,\boldsymbol{\lambda}^i \, , \, \varphi\right)_{\times,ij} + \frac{1}{2}\sum_{\substack{l=0\\ l\neq i}}^{N}\left(\mathsf{R}_{ij}\,\mathsf{R}^{\dagger}_{il}\,\mathsf{R}_{li}\,\mathsf{X}_l\,\boldsymbol{\lambda}^l \, , \, \varphi\right)_{\times,ij} \\
&= \ \frac{1}{2}\left(\mathsf{R}_{ij}\,\boldsymbol{\lambda}^i \, , \, \varphi\right)_{\times,ij} + \frac{1}{2}\left(\mathsf{R}_{ji}\,\mathsf{X}_j\,\boldsymbol{\lambda}^j \, , \, \varphi\right)_{\times,ij}.
\end{aligned} \tag{130}$$

Consequently, $\left[\gamma u\right]_{\Gamma_{ij}} = 0$ for all $\Gamma_{ij} \in \Sigma$ and u is a radiating solution over the entire $\mathbb{R}^d$ space. On the other hand, the strong analytic continuation property for the HTP [26, 28, 37], [49, Section 2], implies $u \equiv 0$ in $\mathbb{R}^d$. Hence, from (127) it follows that the interior Calderón projection is equal to zero all over $\partial\Omega_i$ and

$$\boldsymbol{\lambda}^i \;=\; -\sum_{\substack{j=0\\ j\neq i}}^{N} \widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j, \quad i = 0, \ldots, N. \tag{131}$$

(ii) For the second step of the proof, let us now consider the following radiating solutions on the *complementary domains* Ω_i^c. Since for $N \geq 2$, complementary domains overlap, we follow an argument presented in [49, Section 4.2] and will construct solutions over $(N+1)$ sheets of $\mathbb{R}^d$, with sheets connected through the skeleton Σ (see Fig. 2). One can interpret this as the existence of a *multi-valued* Helmholtz solution over $\mathbb{R}^{d\times(N+1)}$. Let $\mathbb{R}_i^d$ denote the associated sheet to a subdomain Ω_i, $i = 0, \ldots, N$, and define

$$u_i^c \;:=\; \Psi^i(\boldsymbol{\lambda}^i) \quad \text{on } \Omega_i^c \subset \mathbb{R}_i^d\,. \tag{132}$$

Then, we construct the *multi-valued* solution

$$u^{c,\sigma} \;:=\; \left\{\, \sigma_i u_i^c : \mathbb{R}_i^d \mapsto \mathbb{C}, \quad i = 0, \ldots, N \right\}\,, \tag{133}$$

with signs $\sigma_i \in \{\pm 1\}$ independently chosen for the individual subdomains. The superscript σ in $u^{c,\sigma}$ indicates the dependence on these signs. Again, it is our goal to establish $u^{c,\sigma} = 0$. To do so, we appeal to the *strong unique continuation principle* for solutions of the homogeneous Helmholtz equation with piecewise constant wavenumber, see [16, Sect. 3.4.1]:

> *Let the open domain $D \subset \mathbb{R}^d$ be partitioned into open subsets with piecewise smooth boundaries and let $\kappa \in L^\infty(D)$, be piecewise constant with respect to that partition. Then a solution u of $\Delta u + \kappa^2 u = 0$ is uniquely determined by its values on any open ball inside D.*

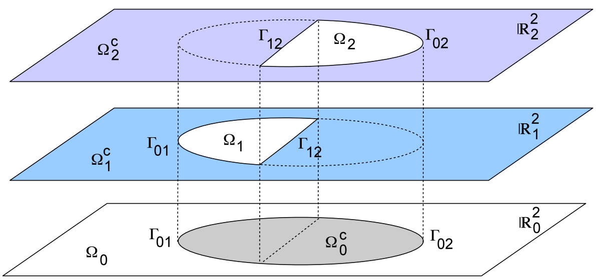


Fig. 2 Multivalued Helmholtz solution in $\mathbb{R}^2$ for $N = 2$

This principle extends to multivalued solutions in a straightforward manner.
To begin with, we take interior traces, compute the trace jump, see (35), over the interface Γ_{ij} and test it with a smooth compactly supported $\varphi \in \widetilde{\widetilde{\mathbf{V}}}_{ij}$:

$$\left(\left[\gamma u^{c,\sigma}\right]_{\Gamma_{ij}}, \varphi\right)_{\times,ij} = \sigma_j \left(\mathsf{R}_{ji}\,\mathsf{X}_j^2\,\gamma^{j,c} u_j^c, \varphi\right)_{\times,ij} - \sigma_i \left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\gamma^{i,c} u_i^c, \varphi\right)_{\times,ij}. \tag{134}$$

By the exterior Calderón projector formula, see (47) and (55), and the integral equations of the multiple traces formulation with $\mathbf{g} = 0$, it holds

$$\gamma^{i,c} u_i^c = \left(-\mathsf{A}_i + \frac{1}{2}\,\mathsf{Id}\right)\boldsymbol{\lambda}^i \overset{(115)}{=} \frac{1}{2}\boldsymbol{\lambda}^i - \frac{1}{2}\sum_{\substack{j=0\\ j\neq i}}^{N} \widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j. \tag{135}$$

Analogous to the derivation of (129) and (130) we can express the terms in (134) as

$$\begin{aligned}
\left(\mathsf{R}_{ji}\,\mathsf{X}_j^2\,\gamma^{j,c} u_j^c, \varphi\right)_{\times,ij} &= \left(\mathsf{R}_{ji}\,\gamma^{j,c}\Psi^j(\boldsymbol{\lambda}^j), \varphi\right)_{\times,ij} \\
&\overset{(135)}{=} \frac{1}{2}\left(\mathsf{R}_{ji}\,\boldsymbol{\lambda}^j, \varphi\right)_{\times,ij} - \frac{1}{2}\sum_{\substack{k=0\\ k\neq j}}^{N} \underbrace{\left(\mathsf{R}_{ji}\,\mathsf{R}_{jk}^{\dagger}\,\mathsf{R}_{kj}\,\mathsf{X}_k\,\boldsymbol{\lambda}^k, \varphi\right)_{\times,ij}}_{=0 \text{ for } k\neq i} \\
&= \frac{1}{2}\left(\mathsf{R}_{ji}\,\boldsymbol{\lambda}^j, \varphi\right)_{\times,ij} - \frac{1}{2}\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i, \varphi\right)_{\times,ij}
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\gamma^{i,c} u_i^c, \varphi\right)_{\times,ij} &= \left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\gamma^{i,c}\Psi^i(\boldsymbol{\lambda}^i), \varphi\right)_{\times,ij} \\
&\overset{(135)}{=} \frac{1}{2}\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i, \varphi\right)_{\times,ij} - \frac{1}{2}\sum_{\substack{k=0\\ k\neq i}}^{N} \underbrace{\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\mathsf{R}_{ik}^{\dagger}\,\mathsf{R}_{ki}\,\mathsf{X}_k\,\boldsymbol{\lambda}^k, \varphi\right)_{\times,ij}}_{=0 \text{ for } k\neq j} \\
&= \frac{1}{2}\left(\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\lambda}^i, \varphi\right)_{\times,ij} - \frac{1}{2}\left(\mathsf{R}_{ji}\,\boldsymbol{\lambda}^j, \varphi\right)_{\times,ij}.
\end{aligned}$$

Thus, (134) is equal to zero, and, consequently, the jump $\left[\gamma u^{c,\sigma}\right]_{\Gamma_{ij}}$ vanishes, if $\sigma_i\sigma_j < 0$. Now we fix $\sigma_0 = +1$ and consider suitable permutations of the remaining signs $\sigma_1, \ldots, \sigma_N$.
If we can select the σ_i such that adjacent subdomains always carry opposite signs we adapt the argument from the case $N = 1$ discussed in Section 3.1.1; the corresponding $u^{c,\sigma}$ will be a global multi-valued homogeneous Helmholtz solution satisfying radiation conditions. Hence, as in [49, Section 4.2] we can conclude that it will vanish.

If such a sign pattern is not possible, we have to rely on the above unique continuation principle. Let us elucidate the reasoning leading to the conclusion $u_i^{c,\sigma} = 0$ for the case $N = 2$ and the situation depicted in Fig. 2. Then three multivalued solutions of the homogeneous Helmholtz equation may be taken into account, listed according to the choice of σ_i in the different subdomains:

σ_1	σ_2		On Γ_{01}	On Γ_{02}	On Γ_{12}
$-$	$-$	$(u^{c,--})$	$u_0^c - \mathsf{X}_0 u_1^c = 0$	$u_0^c - \mathsf{X}_0 u_2^c = 0$	No information
$-$	$+$	$(u^{c,-+})$	$u_0^c - \mathsf{X}_0 u_1^c = 0$	No information	$u_1^c - \mathsf{X}_1 u_2^c = 0$
$+$	$-$	$(u^{c,+-})$	No information	$u_0^c - \mathsf{X}_0 u_2^c = 0$	$u_1^c - \mathsf{X}_1 u_2^c = 0$

The three multivalued solutions may have a non-zero jump across a single interface. In addition, they all agree on Ω_0^c. By the strong continuation principle they must agree everywhere away from the interfaces:

$$u^{c,--} = u^{c,+-} = u^{c,-+} \quad \text{on } \Omega_0^c \cup \Omega_1^c \cup \Omega_2^c \,. \tag{136}$$

By their definition in (133) we conclude

$$u_1^c = -u_1^c \quad \text{and} \quad u_2^c = -u_2^c \quad \Rightarrow \quad u_1^c = u_2^c = 0 \,. \tag{137}$$

Hence, u_0^c can be extended by zero to a homogeneous Helmholtz solution on $\mathbb{R}^d$ and, thus, must vanish, too. These considerations can be generalized to any $N > 2$ in a straightforward fashion. Eventually, (135) implies

$$\boldsymbol{\lambda}^i = \sum_{\substack{j=0 \\ j\neq i}}^{N} \widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j \quad i = 0, \ldots, N \,. \tag{138}$$

Consequently, condition (131) implies $\boldsymbol{\lambda}$ identically equal to zero. □

3.2.7 Coercivity of $\mathbf{M}_N$

We shall first show the following properties:

Lemma 8 *Let* $\boldsymbol{\lambda}^i \in \widetilde{\mathbf{V}}_i$ *and* $\boldsymbol{\lambda}^j \in \widetilde{\mathbf{V}}_j$. *Then, it holds*

$$\left(\boldsymbol{\lambda}^i , \widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j\right)_{\times,i} = -\overline{\left(\boldsymbol{\lambda}^j , \widetilde{\mathsf{X}}_{ji} \boldsymbol{\lambda}^i\right)}_{\times,j}. \tag{139}$$

Proof Let us assume Λ_i non-empty as otherwise the identity holds trivially since $\widetilde{\mathsf{X}}_{ij} \equiv 0$ for all $j = 0, \ldots, N$. Direct expansion of the sesquilinear forms defined over $\partial\Omega_i$ yields

$$\begin{aligned}\left(\boldsymbol{\lambda}^i , \widetilde{\mathsf{X}}_{ij} \boldsymbol{\lambda}^j\right)_{\times,i} &= \left(\boldsymbol{\lambda}^i , \mathsf{R}_{ij}^\dagger \mathsf{R}_{ji} \mathsf{X}_j \boldsymbol{\lambda}^j\right)_{\times,i} = \left(\mathsf{X}_j^\dagger \mathsf{R}_{ji}^\dagger \mathsf{R}_{ij} \boldsymbol{\lambda}^i , \boldsymbol{\lambda}^j\right)_{\times,j} \\ &= -\left(\mathsf{X}_j \mathsf{R}_{ji}^\dagger \mathsf{R}_{ij} \boldsymbol{\lambda}^i , \boldsymbol{\lambda}^j\right)_{\times,j}\end{aligned} \tag{140}$$

now over $\partial\Omega_j$ and where the last step is due to Lemma 1. Due to linearity and to the diagonal structure of the operators, one can easily show that $\mathsf{X}_j \mathsf{R}_{ji}^{\dagger} \mathsf{R}_{ij} = \mathsf{R}_{ji}^{\dagger} \mathsf{R}_{ij} \mathsf{X}_i = \widetilde{\mathsf{X}}_{ji}$ in weak sense. Hence,

$$\left(\boldsymbol{\lambda}^i , \widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j\right)_{\times,i} = -\left(\widetilde{\mathsf{X}}_{ji}\boldsymbol{\lambda}^i , \boldsymbol{\lambda}^j\right)_{\times,j} = -\overline{\left(\boldsymbol{\lambda}^j , \widetilde{\mathsf{X}}_{ji}\boldsymbol{\lambda}^i\right)}_{\times,j} \tag{141}$$

as stated. □

Theorem 10 *The multi-trace formulation of the Helmholtz transmission problem* (115) *is $\mathbb{V}_N$-coercive for $\boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N$, i.e. there exists a constant $\alpha_{\mathbf{M}_N}$ such that*

$$\mathfrak{Re}\left\{\left(\boldsymbol{\lambda} , \left(\mathbf{M}_N + \mathbf{T}_{\mathbf{M}_N}\right) \boldsymbol{\lambda}\right)_{\times}\right\} \geq \alpha_{\mathbf{M}_N} \|\boldsymbol{\lambda}\|^2_{\mathbb{V}_N}, \quad \textit{for all } \boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N, \tag{142}$$

where $\mathbf{T}_{\mathbf{M}_N} : \mathbb{V}_N \to \mathbb{V}_N$ is compact. This result also holds for $\boldsymbol{\lambda} \in \widetilde{\widetilde{\mathbb{V}}}_N \subset \widetilde{\mathbb{V}}_N$.

Proof We need to show that $\mathbf{M}_N$ is positive-definite up to a compact perturbation. Let $\boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N$ and take the skew duality product:

$$(\boldsymbol{\lambda} , \mathbf{M}_N \boldsymbol{\lambda})_{\times} = \sum_{i=0}^{N} \left(\boldsymbol{\lambda}^i , \boldsymbol{\xi}^i\right)_{\times,i} \tag{143}$$

where $\boldsymbol{\xi}^i$ is the i-projection of $\mathbf{M}_N \boldsymbol{\lambda} \in \mathbb{V}_N$. Expansion of the right-hand side yields

$$\sum_{i=0}^{N} \left(\boldsymbol{\lambda}^i , \boldsymbol{\xi}^i\right)_{\times,i} = \sum_{i=0}^{N} \left(\boldsymbol{\lambda}^i , \mathsf{A}_i \boldsymbol{\lambda}^i\right)_{\times,i} - \frac{1}{2} \sum_{i=0}^{N} \sum_{\substack{j=0 \\ j\neq i}}^{N} \left(\boldsymbol{\lambda}^i , \widetilde{\mathsf{X}}_{ij}\boldsymbol{\lambda}^j\right)_{\times,i}. \tag{144}$$

The double sum in (144) is purely imaginary due to Lemma 8. Hence, we obtain

$$\mathfrak{Re}\left\{(\boldsymbol{\lambda} , \mathbf{M}_N \boldsymbol{\lambda})_{\times}\right\} = \sum_{i=0}^{N} \mathfrak{Re}\left\{\left(\boldsymbol{\lambda}^i , \mathsf{A}_i \boldsymbol{\lambda}^i\right)_{\times,i}\right\}, \tag{145}$$

wherein each summand is coercive as shown in Theorem 3 with compact operators $\mathsf{T}_{\mathsf{A}_i}$ and coercivity constants denoted α_{A_i}, $i = 0, \ldots, N$. Then, by taking $\alpha_{\mathbf{M}_N} := \min_{i=0,\ldots,N} \alpha_{\mathsf{A}_i}$ and $\mathbf{T}_{\mathbf{M}_N} := \mathrm{diag}(\mathsf{T}_{\mathsf{A}_i})_{i=0}^{N} : \mathbb{V}_N \to \mathbb{V}_N$, the result follows. □

3.2.8 Existence and stability

We observe a fundamental mismatch beetween the coercivity result of Theorem 10 which refers to the norm of $\mathbb{V}_N$ and the continuity of the sesquilinear form m_N on $\mathbb{V}_N \times \widetilde{\widetilde{\mathbb{V}}}_N$. This defies the standard Fredholm argument [44] and forces us to resort to the more refined stability theory provided by the following two lemmas.

Lemma 9 (Lion's projection lemma [31], Chapter III, Theorem 1.1) *Let H be a Hilbert space and Φ a subspace of H (not necessarily closed). Moreover, let* $\mathsf{b} : H \times \Phi \to \mathbb{R}$ *be a bilinear form satisfying the following properties:*

1. *For every $\varphi \in \Phi$, the linear form $u \mapsto \mathsf{b}(u, \varphi)$ is continuous in H.*
2. *There exists $\alpha > 0$ such that*

$$|\mathsf{b}(\varphi, \varphi)| \geq \alpha \, \|\varphi\|_H^2 \quad \text{for all } \varphi \in \Phi \, . \tag{146}$$

Then for each continuous linear form $l \in H'$, there exists $u_0 \in H$ such that

$$\mathsf{b}(u_0, \varphi) \; = \; \langle l \, , \, \varphi \rangle_H \quad \forall \, \varphi \in \Phi \quad \text{and} \quad \|u_0\|_H \leq \frac{1}{\alpha} \, \|l\|_{H'} \, . \tag{147}$$

Proof We transcript the proof given in [11]. From assumption (1) and the Riesz representation theorem it follows that for every $\varphi \in \Phi$ there exists $\mathsf{B}\,\varphi \in H$ with

$$(u \, , \, \mathsf{B}\,\varphi)_H \; = \; \mathsf{b}(u, \varphi) \quad \forall \, u \in H. \tag{148}$$

This defines a linear (generally unbounded) operator $\mathsf{B} : \Phi \to \Pi := \mathsf{B}(\Phi) \subseteq H$. By assumption (2), the operator B is injective and has an inverse $\mathsf{L} : \Pi \to \Phi$. Again from assumption (2), one has for $p \in \Pi$

$$\|\mathsf{L}\,p\|_H^2 \; \leq \; \frac{1}{\alpha} \mathsf{b}(\mathsf{L}\,p, \mathsf{L}\,p) \; = \; \frac{1}{\alpha} \, (\mathsf{L}\,p \, , \, \mathsf{B}\,\mathsf{L}\,p)_H \; \leq \; \frac{1}{\alpha} \, \|\mathsf{L}\,p\|_H \, \|p\|_H \, , \tag{149}$$

from where $\|\mathsf{L}\,p\|_H \leq \alpha^{-1} \, \|p\|_H$. Hence, L can be extended by continuity to the closure $\overline{\Pi}$ of Π in the H-norm. Let us denote this extension by $\overline{\mathsf{L}}$ satisfying $\overline{\mathsf{L}} : \Pi \to \overline{\Phi}$ where now $\overline{\Phi}$ is a closed subspace of H and thus also Hilbert with norm $\|\cdot\|_H$. Using the Riesz representation theorem on $\overline{\Phi}$ one obtains a $\xi_l \in \overline{\Phi}$ with

$$l(\varphi) \; = \; (\xi_l \, , \, \varphi) \quad \forall \, \varphi \in \overline{\Phi}. \tag{150}$$

Finally, let $\mathsf{P} : H \to \overline{\Pi}$ be the orthogonal projection onto $\overline{\Pi}$, then $u_0 := \mathsf{P}^* \, \overline{\mathsf{L}}^* \xi_l$, where asterisk denote adjoints, satisfies the stated conditions as

$$\begin{aligned} \mathsf{b}(u_0, \varphi) \; &= \; (u_0 \, , \, \mathsf{B}\,\varphi)_H \; = \; \left(\mathsf{P}^* \, \overline{\mathsf{L}}^* \xi_l \, , \, \mathsf{B}\,\varphi\right)_H \; = \; \left(\overline{\mathsf{L}}^* \xi_l \, , \, \mathsf{B}\,\varphi\right)_{\overline{\Pi}} \\ &= \; (\xi_l \, , \, \varphi)_{\overline{\Phi}} \; = \langle l \, , \, \varphi \rangle_H \, , \end{aligned} \tag{151}$$

for all $\varphi \in \Phi$, and

$$\|u_0\|_H \; \leq \; \left\|\mathsf{P}^*\right\|_{H \to H'} \left\|\overline{\mathsf{L}}^* \xi_l\right\|_H \; \leq \; \left\|\overline{\mathsf{L}}\right\|_{H \to H} \|\xi_l\|_H \; \leq \; \alpha^{-1} \, \|l\|_{H'} \, . \tag{152}$$

as stated. □

Remark 11 ([31], Chapter III, Remark 1.1) In general, the above result does not guarantee uniqueness of the solution. The necessary and sufficient condition for this to hold is that Φ is dense in H. Moreover, notice that the solution(s) belong to H and not necessarily to Φ.

Lemma 10 *Let H be a Hilbert space and Φ a subspace of H (not necessarily closed). Moreover, let* $:H \times \Phi \to \mathbb{R}$ *and* $\mathsf{t} : H \times H \to \mathbb{R}$ *be bilinear forms satisfying the following properties:*

1. *For every $\varphi \in \Phi$, the linear form $u \mapsto \mathsf{m}(u, \varphi)$ is continuous in H.*
2. *The linear operator $\mathsf{T} : H \to H$ associated to the bilinear form $\mathsf{t}(\cdot, \cdot)$ is compact and continuous.*
3. *There exists $\alpha > 0$ such that*

$$\mathfrak{Re}\,\{\mathsf{m}(\varphi, \varphi) + \mathsf{t}(\varphi, \varphi)\} \geq \alpha\, \|\varphi\|_H^2\,, \quad \forall\, \varphi \in \Phi. \tag{153}$$

4. *The form $u \mapsto \mathsf{m}(u, \varphi)$ is injective, i.e. $\mathsf{m}(u, \varphi) = 0$ for all $\varphi \in \Phi$, implies $u = 0$.*

Then, for $l \in H'$ there exists $u_0 \in H$ solution of

$$\mathsf{m}(u, \varphi) \;=\; \langle l\,,\, \varphi\rangle\,, \quad \forall\, \varphi \in \Phi, \tag{154}$$

satisfying the stability estimate

$$\|u_0\|_H \leq \frac{\beta_{\mathsf{m}}}{\alpha}\, \|l\|_{H'}\,. \tag{155}$$

where β_{m} is independent of l.

Proof By assumptions (1) and (2), the Riesz theorem applies and there exist an unbounded and a compact operator, M and T, respectively, such that for every $\varphi \in \Phi$, it holds

$$(u\,,\, \mathsf{M}\varphi)_H \;=\; \mathsf{m}(u, \varphi), \quad (u\,,\, \mathsf{T}\varphi)_H \;=\; \mathsf{t}(u, \varphi), \quad \forall\, u \in H, \tag{156}$$

with $\mathsf{M} : \Phi \to H$ and $\mathsf{T} : H \to H$ such that the sesquilinear form induced $\mathsf{B} := \mathsf{M} + \mathsf{T}$ satisfies hypothesis (146) of Lemma 9. Then this lemma applies and there exists a bounded inverse operator $\mathsf{B}^{-1} : H \to H$ with continuity constant α^{-1}. On the other hand, hypothesis (4) implies that M is injective and it can be expressed as

$$\mathsf{M} \;=\; \mathsf{B} - \mathsf{T} \;=\; \left(\mathsf{Id} - \mathsf{T}\,\mathsf{B}^{-1}\right)\mathsf{B} \tag{157}$$

where the operator $\mathsf{T}\,\mathsf{B}^{-1}$ is compact and $\mathsf{Id} - \mathsf{T}\,\mathsf{B}^{-1}$ is injective. Thus, the Fredholm alternative follows and the operator $\mathsf{Id} - \mathsf{T}\,\mathsf{B}^{-1} : H \to H$ has a bounded inverse with continuity constant β_{m}. Consequently, by the Riesz representation for all $l \in \Phi'$, there is a $\xi_l \in \Phi$, such that for $u_0 := \mathsf{M}^{-1}\,\xi_l$, it holds

$$\begin{aligned}
\|u_0\|_H \;=\; \left\|\mathsf{M}^{-1}\,\xi_l\right\|_H &\leq \left\|\mathsf{B}^{-1}\left(\mathsf{Id} - \mathsf{T}\,\mathsf{B}^{-1}\right)^{-1}\right\|_{H\to H} \|\xi_l\|_H \\
&\leq \left\|\mathsf{B}^{-1}\right\|_{H\to H} \left\|\left(\mathsf{Id} - \mathsf{T}\,\mathsf{B}^{-1}\right)^{-1}\right\|_{H\to H} \|\xi_l\|_H \\
&\leq \frac{\beta_{\mathsf{m}}}{\alpha}\, \|l\|_{H'}
\end{aligned} \tag{158}$$

as stated. □

Theorem 11 *There exists a unique solution* $\boldsymbol{\lambda} \in \mathbb{V}_N$ *for the multiple traces formulation* (115) *satisfying*

$$\|\boldsymbol{\lambda}\|_{\mathbb{V}_N} \leq \frac{\beta_{\mathbf{M}_N}}{\alpha_{\mathbf{M}_N}} \|\mathbf{g}\|_{\mathbf{V}_0}. \tag{159}$$

with positive constants $\beta_{\mathbf{M}_N}$ *and* $\alpha_{\mathbf{M}_N}$.

Proof It follows directly by compliance with the assumptions of Lemma 10. In particular, let $H \equiv \mathbb{V}_N$ and $\Phi \equiv \approx{\mathbb{V}}_N$. Our bilinear forms are

$$\mathsf{m}_N : \varphi \mapsto (\mathbf{M}_N \boldsymbol{\lambda}, \varphi)_\times : \Phi \to H \quad \text{and} \quad \mathsf{t}_N : \varphi \mapsto \left(\mathbf{T}_{\mathbf{M}_N} \boldsymbol{\lambda}, \varphi\right)_\times : H \to H,$$

both continuous and such that hypothesis (2) and (3) are satisfied (Theorem 10) with $\alpha_{\mathbf{M}_N}$ given by (142). Moreover, Theorem 9 ensures hypothesis (4) as the operator acting on $\mathbf{g} \in \mathbf{V}_0$ in (115) defines a linear form for all $\varphi \in \approx{\mathbb{V}}_N$ when written as

$$l_{\mathbf{g}}(\varphi) = \frac{1}{2}\left\langle \mathbf{Q}_N \begin{pmatrix} \mathsf{X}_0\, \mathbf{g} \\ -\mathsf{R}_{10}^{\dagger}\, \mathsf{R}_{01}\, \mathbf{g} \\ \vdots \\ -\mathsf{R}_{N0}^{\dagger}\, \mathsf{R}_{0N}\, \mathbf{g} \end{pmatrix}, \varphi \right\rangle$$

Thus, there is a positive constant $\beta_{\mathbf{M}_N}$ independent from $l_{\mathbf{g}}$. □

3.2.9 Mapping properties of $\mathbf{M}_N^{-1}$

Along the lines of the proof of Theorem 6 for $N = 1$, we conclude that the compact operator $\mathbf{T}_{\mathsf{M}_N}$ is in fact smoothing for all N.

Theorem 12 *The operator* $\mathbf{T}_{\mathbf{M}_N}$ *is regularizing, i.e.* $\mathbf{T}_{\mathbf{M}_N} : \mathbb{V}_N \longrightarrow \mathbf{V}_0^{1/2} \times \cdots \times \mathbf{V}_N^{1/2}$.

For $N = 1$, in Theorem 7 we could show that the inverse operator $\mathbf{M}_N^{-1}$ preserves regularity of data. This was done by establishing a relationship between MTF and STF. For $N > 1$ this reduction encounters formidable technical difficulties. The following consideration may give a hint: if $\mathbf{g}_i \in \mathbf{V}_i$, one exchanges test functions $\varphi \in \widetilde{\mathbb{V}}_N$ by those lying in $H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ in the formulation (115). This yields the following expression:

$$\sum_{i=0}^{N} \left(\mathsf{C}_i\, \boldsymbol{\lambda}^i, \varphi^i\right)_{\times,i} = \sum_{i=0}^{N} \left(\mathbf{g}^i, \varphi^i\right)_{\times,i}, \tag{160}$$

which resembles the formulation of Problem 7 when considering φ^i as the oriented restriction of φ on $\partial\Omega^i$. However, then one has to link traces $\boldsymbol{\lambda}$ across Γ_{ij}, which destroys the structure. Therefore, so far we can only conjecture minimal preservation of regularity by the MTF solution operator.

Conjecture 1 (Continuity of $\mathbf{M}_N^{-1}$ in spaces of higher regularity) The inverse MTF operator $\mathbf{M}_N^{-1}$ maps $\mathbf{V}_0^{1/2} \times \cdots \times \mathbf{V}_N^{1/2}$ into itself continuously.

3.2.10 Symmetric multiple trace formulation (sMTF)

In view of the numerical implementation of the proposed method, a variational formulation MTF for which trial and test spaces coincide seems more suitable:

Corollary 2 *If the unique solution* $\boldsymbol{\lambda} \in \mathbb{V}_N$ *of Problem* 6 *belongs to* $\widetilde{\mathbb{V}}_N$, *then it solves the MTF variational problem*

$$\mathsf{m}_N(\boldsymbol{\lambda}, \varphi) \;=\; \frac{1}{2}\left(\begin{pmatrix} \mathsf{X}_0\,\mathbf{g} \\ -\,\mathsf{R}_{10}^{\dagger}\,\mathsf{R}_{01}\,\mathbf{g} \\ \vdots \\ -\,\mathsf{R}_{N0}^{\dagger}\,\mathsf{R}_{0N}\,\mathbf{g} \end{pmatrix}, \varphi\right)_{\times}, \quad \textit{for all } \varphi \in \widetilde{\mathbb{V}}_N\,, \tag{161}$$

with $\mathbf{g} \in \mathbf{V}_0$ *and* m_N *defined as in* (115).

Proof The proof is solely based on the density of $\widetilde{\widetilde{\mathbb{V}}}_N$ in $\widetilde{\mathbb{V}}_N$ and of $\widetilde{\mathbb{V}}_N$ in $\mathbb{V}_N$, see Remark 3, while preserving the dual products. □

4 Discrete formulation

4.1 Preliminaries

We rely on a quasi-uniform family of meshes $\{\Sigma_h, h > 0\}$ of the skeleton Σ. The meshes have to be compatible with the subdomains in the sense that their restrictions $\Sigma_{h\,|\Gamma_{ij}}$ to any interface Γ_{ij} provide a valid family of meshes for Γ_{ij}.

For any interface Γ emdowed with a mesh Γ^h, we write $\mathcal{S}^{-1,0}(\Gamma^h)$ for the space of Γ^h-piecewise constants, and $\mathcal{S}^{0,1}(\Gamma^h)$ for the space of Γ^h-piecewise linear, globally continuous functions, respectively. Moreover, $\mathcal{S}_0^{0,1}(\Gamma^h) \subset \mathcal{S}^{0,1}(\Gamma^h)$ stands for the space of piecewise linear continuous functions that vanish on the boundary $\partial\Gamma$.

We obtain families of finite dimensional spaces $\mathbf{V}_i^h = V_{i,\mathrm{D}}^h \times V_{i,\mathrm{N}}^h$, where $V_{i,\mathrm{D}}^h \equiv \mathcal{S}^{0,1}(\partial\Omega_i^h)$ and $V_{i,\mathrm{N}}^h \equiv \mathcal{S}^{-1,0}(\partial\Omega_i^h)$. They will serve as approximation spaces for the unknown traces associated with a subdomain Ω_i. These spaces are combined into

$$\mathbb{V}_N^h := \mathbf{V}_0^h \times \ldots \times \mathbf{V}_N^h\,, \tag{162}$$

which will serve as discrete trial and test space for a Galerkin discretization of the MTF variational problem (161).

Note that $\mathbb{V}_N^h$ is a subspace of *both* $\mathbb{V}_N$ and $\widetilde{\mathbb{V}}_N$. Hence, in terms of boundary element approximation we can ignore the special trace spaces $\widetilde{\mathbb{V}}_N$ needed for a meaningful variational formulation in Problem 6. Consequently, implementation can rely on standard boundary element spaces and assembly routines from existing codes.

4.1.1 Well-posedness in 2-D

In the following, we establish existence and stability results for discrete solutions in two dimensions as the three-dimensional case is more technically involved. In 2-D the meshes Σ_h boil down to partitionings of Σ into line segments. All triple points, see Fig. 1, have to be mesh points. The index h will also denote the meshwidth.

So-called *dual meshes* will be instrumental in the proofs: for an open or closed polygon Γ equipped with a mesh Γ^h the associated dual mesh $\hat{\Gamma}^h$ is the partitioning of Γ defined by the midpoints of the line segments of Γ^h, and, possibly, the endpoints of Γ. Piecewise constants on Γ^h and continuous piecewise linear functions on $\hat{\Gamma}^h$ that vanish at the endpoints of Γ enjoy L^2-duality in the following sense [35, Section 4]:

$$\sup_{0\neq v_h\in\mathcal{S}_0^{0,1}(\hat{\Gamma}^h)} \frac{|\langle\psi_h\,,\,v_h\rangle|}{\|v_h\|_{L^2(\Gamma)}} \geq C_{\mathrm{ST}}\,\|\psi_h\|_{L^2(\Gamma)}\,, \quad \forall\, h>0, \tag{163}$$

for all $\psi_h\in\mathcal{S}^{-1,0}(\Gamma^h)$. Of course, here and below the duality pairing could be replaced with the inner product $(\cdot\,,\,\cdot)_{L^2(\Gamma)}$. As an immediate consequence of (163) we have the dual estimate

$$\sup_{\substack{\psi_h\in\mathcal{S}^{-1,0}(\Gamma^h)\\ \psi_h\neq 0}} \frac{|\langle\psi_h\,,\,v_h\rangle|}{\|\psi_h\|_{L^2(\Gamma)}} \geq C_{\mathrm{ST}}\,\|v_h\|_{L^2(\Gamma)} \quad \forall v_h\in\mathcal{S}_0^{0,1}(\hat{\Gamma}^h)\,, \quad \forall h>0. \tag{164}$$

This will used to establish inverse inequalities between $\widetilde{H}^{-1/2}(\Gamma_{ij})$ and $H^{-1/2}(\Gamma_{ij})$.

Lemma 11 *Given a quasi-uniform family of meshes $\{\Gamma^h\}_{h>0}$ for a polygon Γ, the following inverse estimate*

$$\|\varphi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C_1(1+|\log h|)\,\|\varphi_h\|_{H^{-1/2}(\Gamma)} \tag{165}$$

holds true for all piecewise constants $\varphi_h\in\mathcal{S}^{-1,0}(\Gamma^h)$ and with C_1 independent of the meshwidth $h>0$.

Proof Denote by $\hat{\Gamma}^h$ the dual mesh of Γ_h as introduced above. From Mclean and Steinbach [35, Sect. 4, proof of Thm. 4.1], it is known that the following inverse inequality holds true

$$\|u_h\|_{\widetilde{H}^{1/2}(\Gamma)} \leq C_{\mathrm{MS}}(1+|\log h|)\,\|u_h\|_{H^{1/2}(\Gamma)} \quad , \quad \forall h>0, \tag{166}$$

for all $u_h\in\mathcal{S}_0^{0,1}(\hat{\Gamma}^h)$. Appealing to (163), we define a linear projection $\mathsf{J}_h^1: L^2(\Gamma)\to\mathcal{S}_0^{0,1}(\hat{\Gamma}^h)$ through

$$\left\langle\varphi_h\,,\,\mathsf{J}_h^1 u\right\rangle = \langle\varphi_h\,,\,u\rangle\,, \quad \forall\,\varphi_h\in\mathcal{S}^{-1,0}(\Gamma^h)\,, \tag{167}$$

which satisfies

$$\left\| \mathsf{J}_h^1 u \right\|_{L^2(\Gamma)} \leq C_{\mathrm{ST}}^{-1} \|u\|_{L^2(\Gamma)}, \quad \forall h > 0 . \tag{168}$$

Indeed, from (164) and setting $v_h := \mathsf{J}_h^1 u \in \mathcal{S}_0^{0,1}(\hat{\Gamma}^h)$ in (164), we get

$$\begin{aligned} C_{\mathrm{ST}} \left\| \mathsf{J}_h^1 u \right\|_{L^2(\Gamma)} &\leq \sup_{0 \neq \psi_h \in \mathcal{S}^{-1,0}(\Gamma^h)} \frac{\left| \left\langle \psi_h , \mathsf{J}_h^1 u \right\rangle \right|}{\|\psi_h\|_{L^2(\Gamma)}} \\ &\leq \sup_{0 \neq \psi_h \in \mathcal{S}^{-1,0}(\Gamma^h)} \frac{|\langle \psi_h , u \rangle|}{\|\psi_h\|_{L^2(\Gamma)}} \leq \|u\|_{L^2(\Gamma)}, \end{aligned} \tag{169}$$

from which we infer (168). We also make use of the $L^2(\Gamma)$-orthogonal projector $\mathsf{P}_h : L^2(\Gamma) \mapsto \mathcal{S}_0^{0,1}(\hat{\Gamma}^h)$, defined by

$$\langle \mathsf{P}_h f , v_h \rangle = \langle f , v_h \rangle, \quad \forall v_h \in \mathcal{S}_0^{0,1}(\hat{\Gamma}^h) \tag{170}$$

In [46, Section 10.2] we find the projection error estimate

$$\| f - \mathsf{P}_h f \|_{L^2(\Gamma)} \leq C_2 \| f \|_{H^1(\Gamma)} \quad \forall f \in H^1(\Gamma), \quad \forall h > 0 . \tag{171}$$

From [35, Theorem 4.2] we learn that this projector is also continuous in $H^1(\Gamma)$:

$$\|\mathsf{P}_h f\|_{H^1(\Gamma)} \leq C_{\mathsf{P}} \| f \|_{H^1(\Gamma)} \quad \forall f \in H^1(\Gamma), \quad \forall h > 0 . \tag{172}$$

Hence, for all $h > 0$ and $u \in H^1(\Gamma)$, we find

$$\begin{aligned} \left\| \mathsf{J}_h^1 u \right\|_{H^1(\Gamma)} &\leq \left\| \mathsf{J}_h^1 u - \mathsf{P}_h u \right\|_{H^1(\Gamma)} + \|\mathsf{P}_h u\|_{H^1(\Gamma)} \\ \text{(172) \& [43, Thm. 4.4.3]} &\leq C_1 h^{-1} \left\| \mathsf{J}_h^1 u - \mathsf{P}_h u \right\|_{L^2(\Gamma)} + C_{\mathsf{P}} \|u\|_{H^1(\Gamma)} \\ &\leq C_1 h^{-1} \left\| \mathsf{J}_h^1 (\mathsf{Id} - \mathsf{P}_h) u \right\|_{L^2(\Gamma)} + C_{\mathsf{P}} \|u\|_{H^1(\Gamma)} \\ \text{(168)} &\leq C_1 h^{-1} C_{\mathrm{ST}}^{-1} \|(\mathsf{Id} - \mathsf{P}_h) u\|_{L^2(\Gamma)} + C_{\mathsf{P}} \|u\|_{H^1(\Gamma)} \\ &\leq C_1 h^{-1} C_{\mathrm{ST}}^{-1} C_2 h \|u\|_{H^1(\Gamma)} + C_{\mathsf{P}} \|u\|_{H^1(\Gamma)} \\ &\leq \widetilde{C}_1 \|u\|_{H^1(\Gamma)} \end{aligned}$$

with $\widetilde{C}_1 := C_1 C_{\mathrm{ST}}^{-1} C_2 + C_{\mathsf{P}}$. Then, in light of (168) the interpolation between $H^1(\Gamma)$ and $L^2(\Gamma)$ yields, with $C_3 = C_{\mathrm{ST}}^{-1/2} \widetilde{C}_1^{1/2}$,

$$\left\| \mathsf{J}_h^1 u \right\|_{H^{1/2}(\Gamma)} \leq C_3 \|u\|_{H^{1/2}(\Gamma)} \quad \forall u \in H^{1/2}(\Gamma), \quad \forall h > 0. \tag{173}$$

Finally, we obtain the desired bound as follows. Let $\varphi_h \in \mathcal{S}^{-1,0}(\Gamma^h)$, then by definition of the dual norm in (13)

$$\begin{aligned}
\|\varphi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} &= \sup_{0\neq v\in H^{1/2}(\Gamma)} \frac{|\langle \varphi_h\,,\, v\rangle|}{\|v\|_{H^{1/2}(\Gamma)}} \\
(173),\ (167) &\leq C_3 \sup_{0\neq v\in H^{1/2}(\Gamma)} \frac{\left|\left\langle \varphi_h\,,\, \mathsf{J}_h^1\, v\right\rangle\right|}{\left\|\mathsf{J}_h^1\, v\right\|_{H^{1/2}(\Gamma)}} \\
&= C_3 \sup_{0\neq v_h\in \mathcal{S}_0^{0,1}(\hat{\Gamma}^h)} \frac{|\langle \varphi_h\,,\, v_h\rangle|}{\|v_h\|_{H^{1/2}(\Gamma)}} \\
(166) &\leq C_3 C_{\mathrm{MS}}(1+|\log h|) \sup_{0\neq v_h\in \mathcal{S}_0^{0,1}(\hat{\Gamma}^h)} \frac{|\langle \varphi_h\,,\, v_h\rangle|}{\|v_h\|_{\widetilde{H}^{1/2}(\Gamma)}} \\
&\leq C_3 C_{\mathrm{MS}}(1+|\log h|) \sup_{0\neq v\in \widetilde{H}^{1/2}(\Gamma)} \frac{|\langle \varphi_h\,,\, v\rangle|}{\|v\|_{\widetilde{H}^{1/2}(\Gamma)}} \qquad (174)
\end{aligned}$$

and the result follows from the definition of the norm of $H^{-1/2}(\Gamma)$, see (13).

□

Corollary 3 *In two dimensions we have the h-uniform inverse estimate*

$$\|\varphi_h\|_{\widetilde{\mathbb{V}}_N} \;\leq\; C_N(1+|\log h|)\, \|\varphi_h\|_{\mathbb{V}_N} \qquad \forall \varphi_h \in \widetilde{\mathbb{V}}_N^h\,, \qquad \forall h>0\,. \qquad (175)$$

Proof The norms in $\mathbb{V}_N$ (resp. in $\widetilde{\mathbb{V}}_N$) are given as the sum of norms for each component $\boldsymbol{\varphi}_h^i$ in $\mathbf{V}_i^h$ (resp. in $\widetilde{\mathbf{V}}_i^h$), $i=0,\ldots,N$. Thus, we need to prove

$$\left\|\varphi_h^i\right\|_{\widetilde{\mathbf{V}}_i} \;\leq\; C_i(1+|\log h|)\, \left\|\varphi_h^i\right\|_{\mathbf{V}_i}, \qquad C_i>0,\; i=0,\ldots,N. \qquad (176)$$

Recall that $\mathbf{V}_i^h$ and $\widetilde{\mathbf{V}}_i^h$ themselves are cartesian products of finite-dimensional conforming Dirichlet and Neumann trace spaces: $\mathbf{V}_i^h \subset \mathbf{V}_i := H^{1/2}(\partial\Omega_i) \times H^{-1/2}(\partial\Omega_i)$ and $\widetilde{\mathbf{V}}_i^h \subset \widetilde{\mathbf{V}}_i := H^{1/2}(\partial\Omega_i) \times \widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i)$. Since the only difference between spaces occurs on their Neumann components, showing (176) is equivalent to prove

$$\left\|\varphi_{h,\mathrm{N}}^i\right\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i^h)} \;\leq\; C_i(1+|\log h|)\, \left\|\varphi_{h,\mathrm{N}}^i\right\|_{H^{-1/2}(\partial\Omega_i^h)} \qquad (177)$$

Now, by definition of $\widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i^h)$, the norm is obtained as a sum of individual norms on each interface Γ_{ij}^h, $j\in\Lambda_i$. By Lemma 11, it holds

$$\begin{aligned}
\left\|\varphi_{h,\mathrm{N}}^i\right\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\partial\Omega_i^h)} &= \sum_{j\in\Lambda_i} \left\|\mathsf{R}_{ij}^{\mathrm{N},h}\, \varphi_{h,\mathrm{N}}^i\right\|_{\widetilde{H}^{-1/2}(\Gamma_{ij}^h)} \\
&\leq (1+|\log h|) \sum_{j\in\Lambda_i} C_{ij} \left\|\mathsf{R}_{ij}^{\mathrm{N},h}\, \varphi_{h,\mathrm{N}}^i\right\|_{H^{-1/2}(\Gamma_{ij}^h)} \\
&\leq C_i(1+|\log h|)\, \left\|\varphi_{h,\mathrm{N}}^i\right\|_{H^{-1/2}(\partial\Omega_i^h)} \qquad (178)
\end{aligned}$$

where $\mathsf{R}_{ij}^{\mathrm{N},h}$ denotes the restriction operator over Γ_{ij}^h for Neumann data and $C_i = \max_{j\in\Lambda_i}\{C_{ij}\}$. □

Theorem 13 *Under Conjecture* 1*, in two dimensions and for quasi-uniform families of skeleton meshes as introduced above, we can find* $h_0 > 0$ *such that the following discrete inf-sup condition holds*

$$\sup_{\varphi_h\in\mathbb{V}_N^h} \frac{|\mathsf{m}_N(\boldsymbol{\lambda}_h,\varphi_h)|}{\|\varphi_h\|_{\mathbb{V}_N}} \;\geq\; \alpha_{\mathbf{M}_N}\,\|\boldsymbol{\lambda}_h\|_{\mathbb{V}_N}\,,\quad \forall\,\boldsymbol{\lambda}_h\in\mathbb{V}_N^h\,,\quad \forall h<h_0\,. \tag{179}$$

Proof We reuse the notations from the proof of Theorem 11. Denote by $\mathbf{P}_h : \widetilde{\mathbb{V}}_N \to \mathbb{V}_N^h$ the orthogonal projector onto the discrete space that realizes best approximation in the $\widetilde{\mathbb{V}}_N$-norm. Fix any $\boldsymbol{\lambda}_h\in\mathbb{V}_N^h$. Using that $\mathbf{M}_N$ is bounded and invertible on $\mathbb{V}_N$, we can define $\varphi_h := \mathbf{P}_h\left(\mathsf{Id}+\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\right)\boldsymbol{\lambda}_h$ in $\mathbb{V}_N^h$.

Then,

$$\begin{aligned}
\mathsf{m}_N(\boldsymbol{\lambda}_h,\varphi_h) &= \mathsf{m}_N(\boldsymbol{\lambda}_h,\mathbf{P}_h\left(\mathsf{Id}+\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\right)\boldsymbol{\lambda}_h)\\
&= \mathsf{m}_N(\boldsymbol{\lambda}_h,\boldsymbol{\lambda}_h)+\mathsf{m}_N(\boldsymbol{\lambda}_h,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)\\
&\quad +\mathsf{m}_N(\boldsymbol{\lambda}_h,(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)\\
&= \mathsf{m}_N(\boldsymbol{\lambda}_h,\boldsymbol{\lambda}_h)+\mathsf{t}_N(\boldsymbol{\lambda}_h,\boldsymbol{\lambda}_h)+\mathsf{m}_N(\boldsymbol{\lambda}_h,(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)\\
&= \mathsf{b}_N(\boldsymbol{\lambda}_h,\boldsymbol{\lambda}_h)+\mathsf{m}_N(\boldsymbol{\lambda}_h,(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)
\end{aligned}$$

By coercivity of $\mathsf{b}_N(\cdot,\cdot) := \mathsf{m}_N(\cdot,\cdot)+\mathsf{t}_N(\cdot,\cdot)$, we obtain by Theorems 10 and 8,

$$\begin{aligned}
|\mathsf{m}_N(\boldsymbol{\lambda}_h,\varphi_h)| \;&\geq\; |\mathsf{b}_N(\boldsymbol{\lambda}_h,\boldsymbol{\lambda}_h)| - \left|\mathsf{m}_N(\boldsymbol{\lambda}_h,(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)\right|\\
&\geq\; \alpha_{\mathbf{M}_N}\,\|\boldsymbol{\lambda}_h\|^2_{\mathbb{V}_N} - C_{\mathsf{m}_N}\,\|\boldsymbol{\lambda}_h\|_{\widetilde{\mathbb{V}}_N}\,\left\|(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h)\right\|_{\widetilde{\mathbb{V}}_N}
\end{aligned}$$

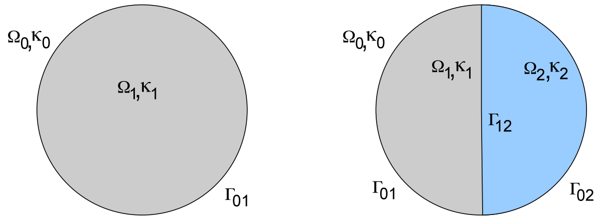
where C_{m_N} is the continuity constant of the bilinear form m_N. Thus, we must show that the the last term tends to zero as $h\to 0$. The operator $\mathbf{T}_{\mathbf{M}_N} : \mathbb{V}_N \to \mathbb{V}_N$ is compact and smoothing acording to Theorem 12, while $\mathbf{M}_N^{-1}$ is the solution operator for the continuous variational problem conjectured to possess the mapping properties detailed in Conjecture 1. Thus, $\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h$ belongs to $\mathbf{V}_0^{1/2}\times\cdots\times\mathbf{V}_N^{1/2}$. By this extra regularity, for some $0<\epsilon<\frac{1}{2}$ and $\boldsymbol{\lambda}$ in $\mathbf{V}_0^{1/2}\times\cdots\times\mathbf{V}_N^{1/2}$, it holds

$$\inf_{\varphi_h\in\mathbb{V}_N^h}\|\boldsymbol{\lambda}-\varphi_h\|_{\widetilde{\mathbb{V}}_N} \;\leq\; C(\epsilon)\inf_{\varphi_h\in\mathbb{V}_N^h}\left\|\boldsymbol{\lambda}-\boldsymbol{\varphi}_h\right\|_{\prod_{i=0}^N H^{1/2}(\partial\Omega_i)\times H^{-1/2+\epsilon}(\partial\Omega_i)} \tag{180}$$

for $C(\epsilon)>0$, due to the continuous embedding $\widetilde{H}^{-1/2+\epsilon}(\Gamma_{ij}) = H^{-1/2+\epsilon}(\Gamma_{ij}) \subset \widetilde{H}^{-1/2}(\Gamma_{ij})$. Appealing to best approximation estimates from [46, Section 10.2] we arrive at

$$\left\|(\mathbf{P}_h-\mathsf{Id})\,\mathbf{M}_N^{-1}\,\mathbf{T}_{\mathbf{M}_N}\,\boldsymbol{\lambda}_h\right\|_{\widetilde{\mathbb{V}}_N} \;=\; C_{\mathbf{P}}(\epsilon)h^{1/2-\epsilon}\,\|\boldsymbol{\lambda}_h\|_{\mathbb{V}_N}\,, \tag{181}$$

Fig. 3 Test geometries for $N = 1$ and $N = 2$ scatterers. Circle radius equals to one

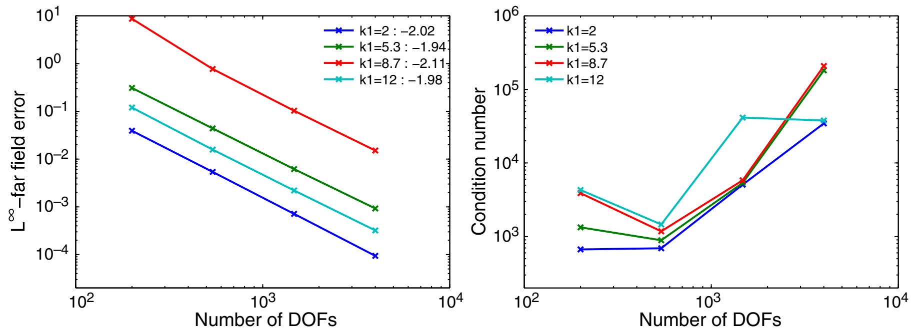


for $C_{\mathbf{P}}(\epsilon) > 0$ that does neither depend on h nor on $\boldsymbol{\lambda}_h$. Now, using the inverse estimate from Lemma 3 valid for 2-D, we finally obtain

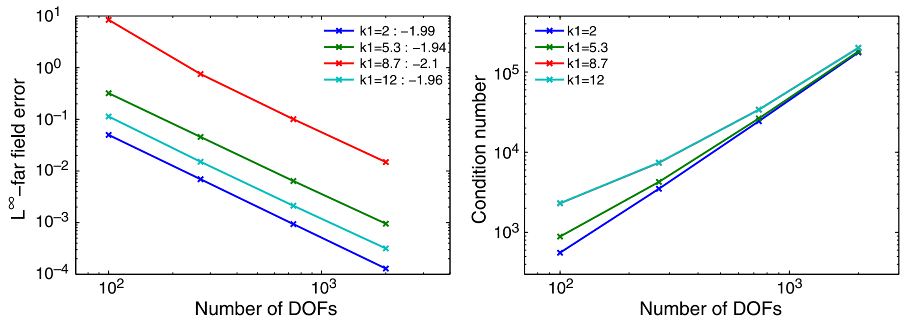
$$|\mathsf{m}_N(\boldsymbol{\lambda}_h, \varphi_h)| \;\geq\; \alpha_{\mathbf{M}_N} \|\boldsymbol{\lambda}_h\|^2_{\mathbb{V}_N} - C_{\mathsf{m}_N} C_{\mathbf{P}}(\epsilon) C_N h^{1/2-\epsilon}(1 + |\log h|) \|\boldsymbol{\lambda}_h\|^2_{\mathbb{V}_N} \quad (182)$$

so that the factor in front of the second norm tends to zero when h does. Therefore, it can be pushed below $\frac{1}{2}\alpha_{\mathbf{M}_N}$ by making h sufficiently small. □

By standard arguments, Theorem 13 implies asymptotic quasioptimality of Galerkin boundary element solutions in the $\mathbb{V}_N$ trace norm.

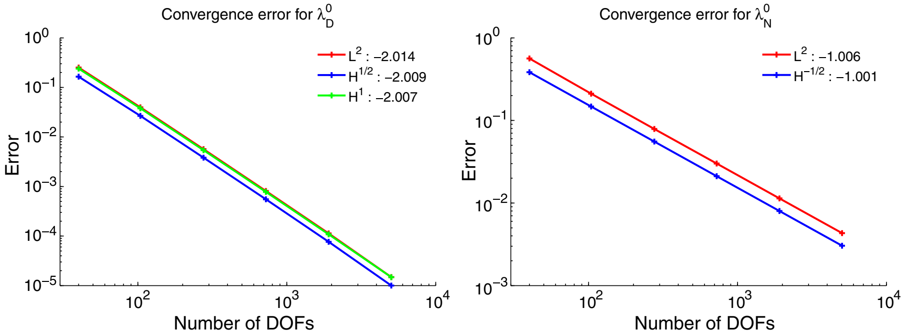


(a) Multiple Traces Formulation ($N = 1$)

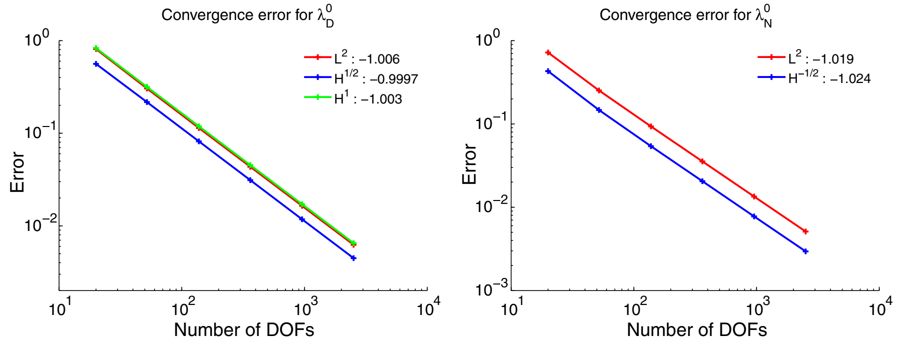


(b) Single Traces Formulation ($N = 1$)

Fig. 4 Conditioning numbers and far-field L^{∞}-error convergence as $\mathcal{O}(h^2)$ for the MTF and STF of a single scatterer (a circle of radius one) for different values of κ_1 with $\kappa_0 = 1$ and for an impinging plane wave with angle $\theta = 0$. Exact far-field solution is obtained via Mie series



(a) Multiple Traces Formulation ($N = 1$)



(b) Single Traces Formulation ($N = 1$)

Fig. 5 Error convergence rates in different norms for the exterior Dirichlet and Neumann traces $(\lambda_D^0, \lambda_N^0)$ on a single circular scatterer of radius one with $\kappa_1 = 2$ with $\kappa_0 = 1$ for an impinging plane wave with angle $\theta = 0$. Errors are obtained with respect to the linear interpolation of the exact Mie series solution

Remark 12 A similar result in $\mathbb{R}^3$ will hold, provided there exists an equivalent of the inverse inequality (Lemma 3) on surfaces.

5 Numerical results

We now present numerical simulations for two-dimensional canonic scatterers ($N = 1$ and $N = 2$) for which analytic solutions are available in the form of Mie series [50, Section 3.1.5]. The case of two subdomains already contains all the difficulties pertaining Lipschitz domains with sharp corners.[1]

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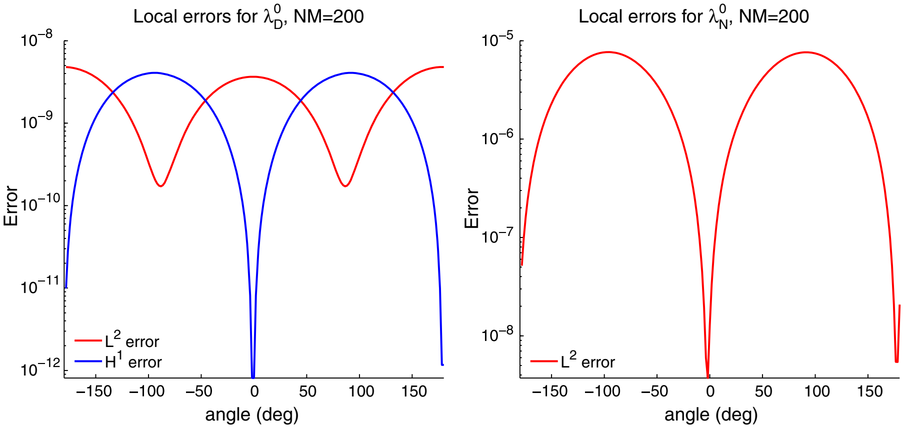
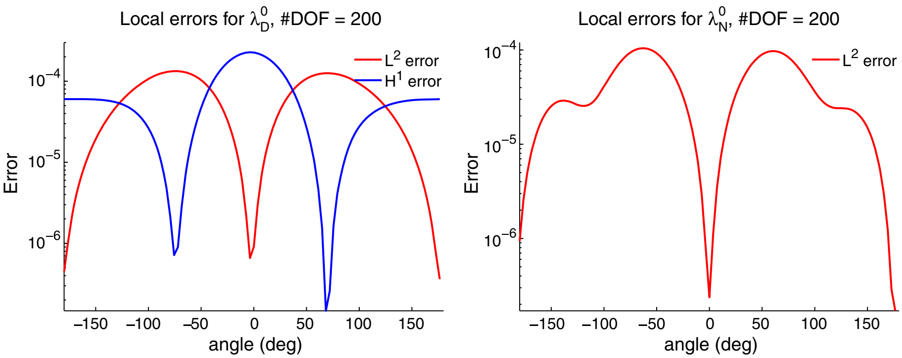
[1]Experiments were performed on MATLAB 2009b, 32-bit, running on a MACBOOK PRO with 2.93 GHz, 4 GB RAM, and based on the 2-D boundary element MEX-FORTRAN library developed by A. Bendali.

5.1 Single subdomain ($N = 1$)

We analyze one penetrable obstacle consisting of a circle of radius one (see Fig. 3) with varying interior wavenumber κ_1, cf. Section 3.1. In the exterior domain we set a wavenumber $\kappa_0 = 1$ and the exact solution for the configuration is provided by means of Mie series. In Fig. 4, we present L^∞-error convergence rates for the far-field obtained and conditioning numbers for the MTF and STF. We observe a convergence rate of $\mathcal{O}(h^2)$ independently of the values of κ_1 for uniformly refined meshes, where h is inversely dependent on the number of unknowns. This is due to the smoothing properties of the far-field operator. The numerical computation of integral operators requires h to be smaller than a value dependent on κ_1 and which must decrease as the wavenumber κ_1 increases.



(a) Multiple Traces Formulation ($N = 1$)

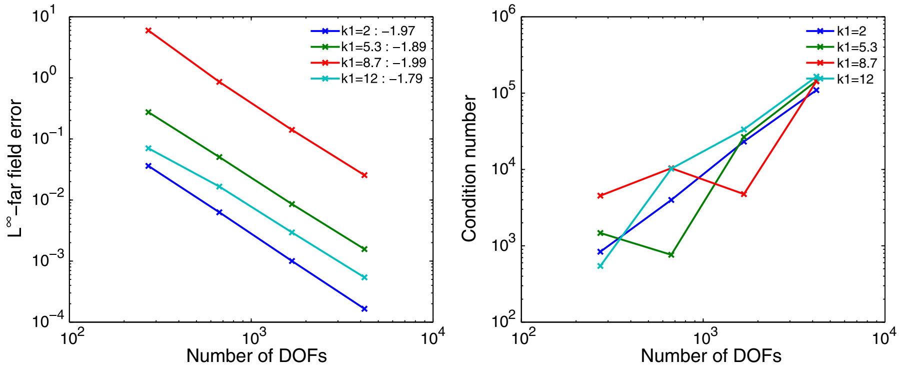


(b) Single Traces Formulation ($N = 1$)

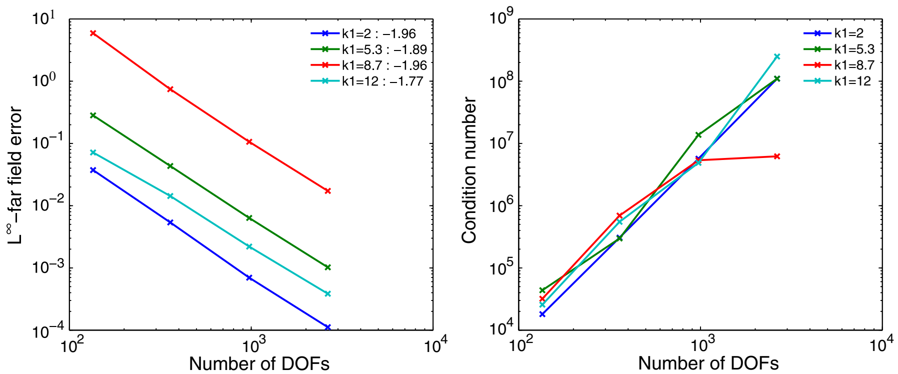
Fig. 6 MTF and STF local errors in different norms for the exterior Dirichlet and Neumann traces (λ_D^0, λ_N^0) on a single circular scatterer ($N = 1$) of radius one with $\kappa_1 = 2$ with $\kappa_0 = 1$ for an impinging plane wave with angle $\theta = 0$. A total of 200 degrees of freedom is used and errors are obtained with respect to the linear interpolation of the exact Mie series solution

More interesting is the comparison between both MTF and STF on their convergence rates for Dirichlet and Neumann traces on $\partial\Omega$. Figure 5 depicts convergence rates in different norms taken with respect to the piecewise linear interpolation of the exact solution over the mesh. Since we obtain similar convergence rates for all norms this points towards the interpolation error itself as the largest contribution. Specifically, one should observe a decay as in $\mathcal{O}(h)$ for the H^1-norm but instead we obtain $\mathcal{O}(h^2)$ due to the regularity of the right-hand side.

In Fig. 6, one can observe the distribution of local trace errors for both formulations. Observe the difference in order of magnitude (10^4) between the local error for Dirichlet data in MTF and STF. On the other hand, Neumann local errors are only slightly better in the MTF. This will serve as comparison for the case of multiple subdomains.



(a) Multiple Traces Formulation ($N = 2$)



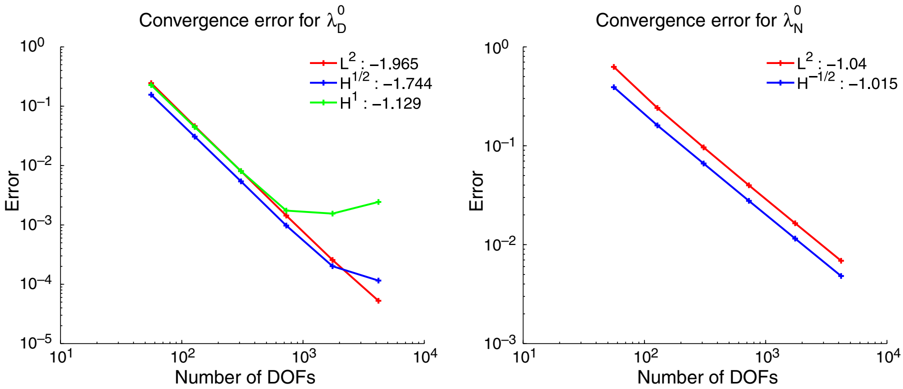
(b) Single Traces Formulation (N =2)

Fig. 7 Far-field error convergence as $\mathcal{O}(h^2)$ for two scatterers (defining a circle of radius one) for different values of $\kappa_1 = \kappa_2$ with $\kappa_0 = 1$ and for an impinging plane wave with angle $\theta = 0$. Exact far-field solution is obtained via Mie series

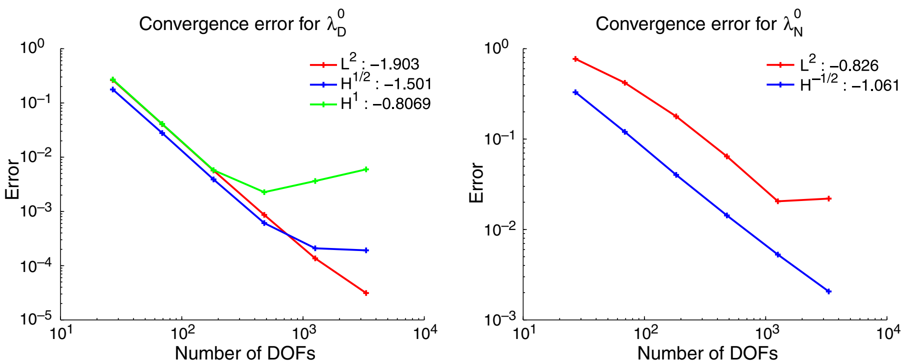
5.2 Two subdomains ($N = 2$)

We now consider the second geometric configuration in Fig. 3, i.e. two half-circles of radius one joined at the polar diameter. Left and right hand sides are described by wavenumbers κ_1 and κ_2, respectively. As before, κ_0 is all along set to one.

In this case, one can compare with the Mie series exact solution to validate the model. Figure 7 shows far-field L^∞-error convergence rates and conditioning numbers for the solutions obtained via the MTF and STF. Uniform mesh refinement yields once more expected convergence rate of $\mathcal{O}(h^2)$ independently from values of κ_1. For different mesh refinements, Figs. 8, 9 and 10 depict MTF and STF L^2-, $H^{1/2}$- and H^1-error norms for Dirichlet traces and in L^2- and $H^{-1/2}$-norms for Neumann traces for a fixed $\kappa_1 = \kappa_2$. Again, errors

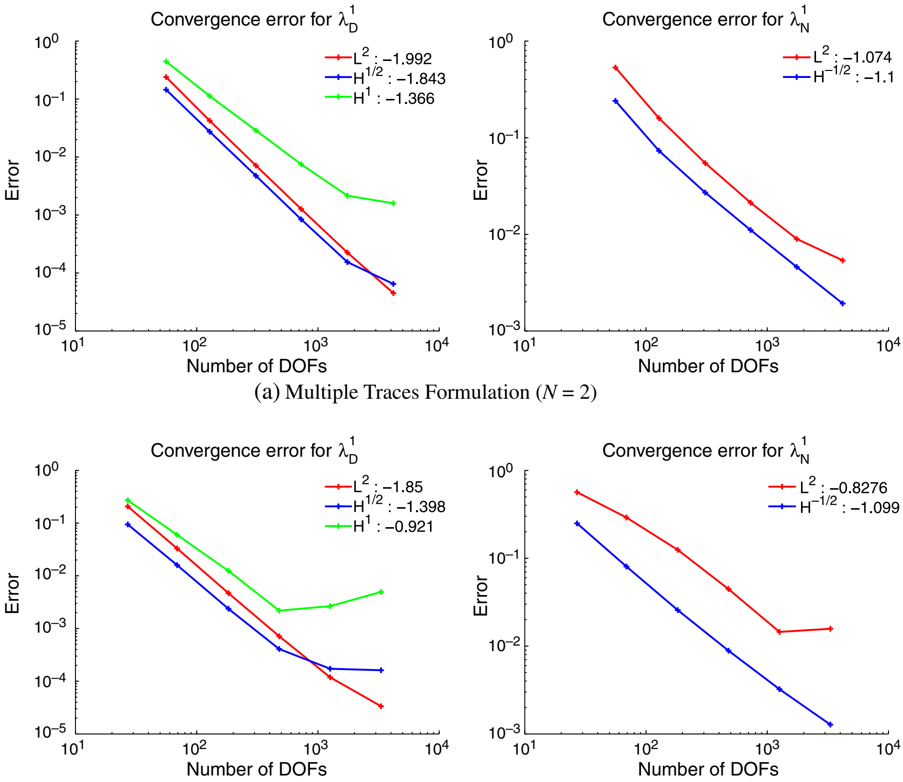


(a) Multiple Traces Formulation ($N = 2$)



(b) Single Traces Formulation ($N = 2$)

Fig. 8 Error convergence rates in different norms for the Dirichlet and Neumann traces $(\lambda_D^0, \lambda_N^0)$ on $\partial\Omega_0$ for a split circle of radius one ($N = 2$) as described in Fig. 3 for an impinging plane wave with angle $\theta = 0$. Parameters $\kappa_1 = \kappa_2 = 2$ and $\kappa_0 = 1$. Errors are obtained with respect to the linear interpolation of the exact Mie series solution



(a) Multiple Traces Formulation ($N = 2$)

(b) Single Traces Formulation ($N = 2$)

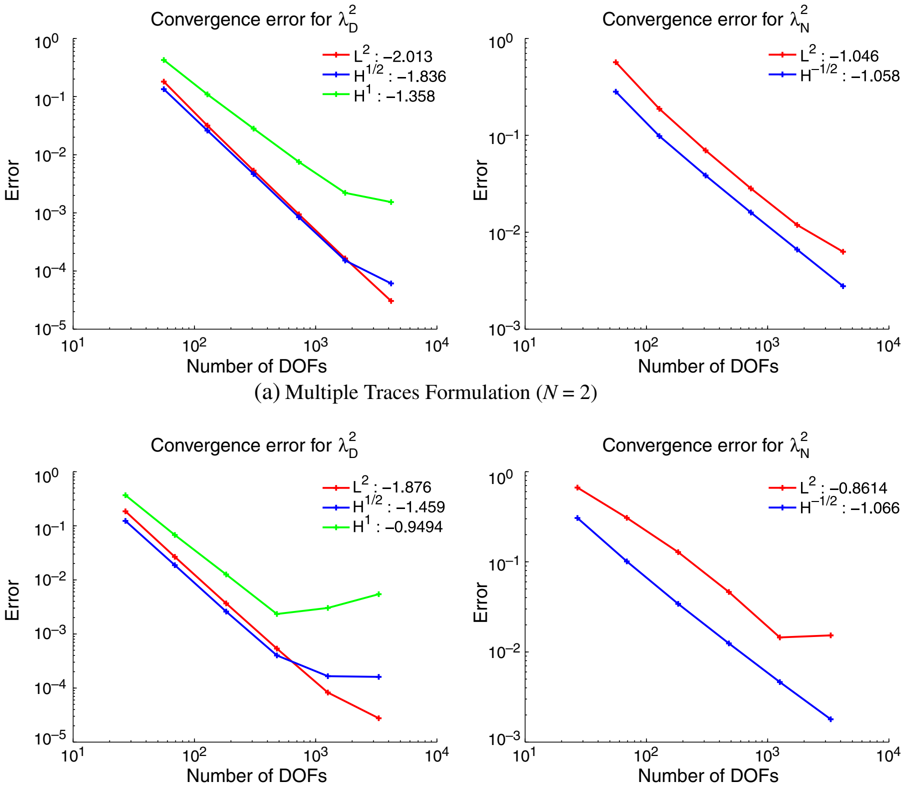
Fig. 9 Error convergence rates in different norms for the Dirichlet and Neumann traces $(\lambda_D^1, \lambda_N^1)$ on $\partial\Omega_1$ for a split circle of radius one ($N = 2$) as described in Fig. 3 for an impinging plane wave with angle $\theta = 0$. Parameters $\kappa_1 = \kappa_2 = 2$ and $\kappa_0 = 1$. Errors are obtained with respect to the linear interpolation of the exact Mie series solution

are taken in absolute terms with respect to the far-field and traces derived from Mie series.

Figure 11 reveals the loss in H^1-regularity at triple points—angles 90° and −90°—when compared to the local L^2-norm. This is explained by the stronger measure of regularity which cannot be achieved by the discretization bases, in particular at triple points. Notice that this also occurs for the single trace formulation.

5.3 Preconditioning

As observed, an increase in the number of unknowns rapidly degrades the conditioning number of the associated matrix and, consequently, the GMRES



(a) Multiple Traces Formulation ($N = 2$)

(b) Single Traces Formulation ($N = 2$)

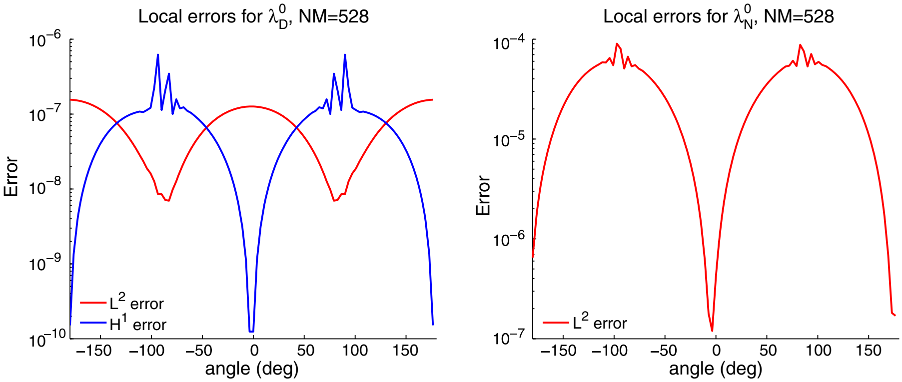
Fig. 10 Error convergence rates in different norms for the Dirichlet and Neumann traces $(\lambda_D^2, \lambda_N^2)$ on $\partial\Omega_2$ for a split circle of radius one ($N = 2$) as described in Fig. 3 for an impinging plane wave with angle $\theta = 0$. Parameters $\kappa_1 = \kappa_2 = 2$ and $\kappa_0 = 1$. Errors are obtained with respect to the linear interpolation of the exact Mie series solution

algorithm requires more iterations to converge [41]. As a preconditioner, we use of the block diagonal operator:

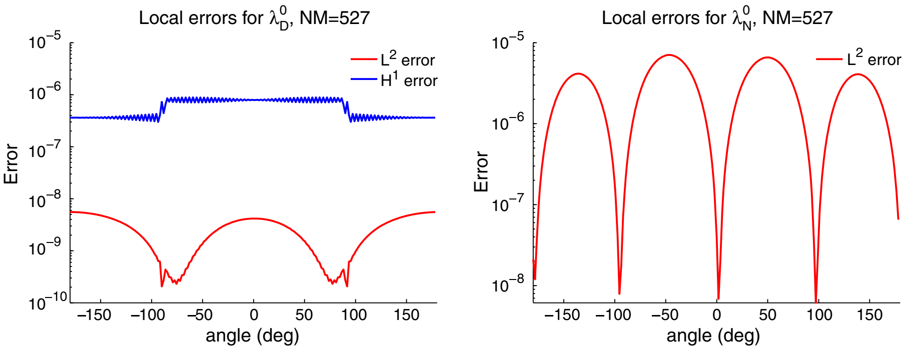
$$\mathbf{Z} := \begin{pmatrix} \mathsf{A}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathsf{A}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathsf{A}_N \end{pmatrix} \tag{183}$$

whose discrete form is denoted by $\mathbf{Z}_{N,h}$.

For a scatterer composed of two semicircles of radius one, Figs. 12 and 13 show relative residual convergence rates up to a tolerance of `tol`=1e-10 for



(a) Multiple Traces Formulation

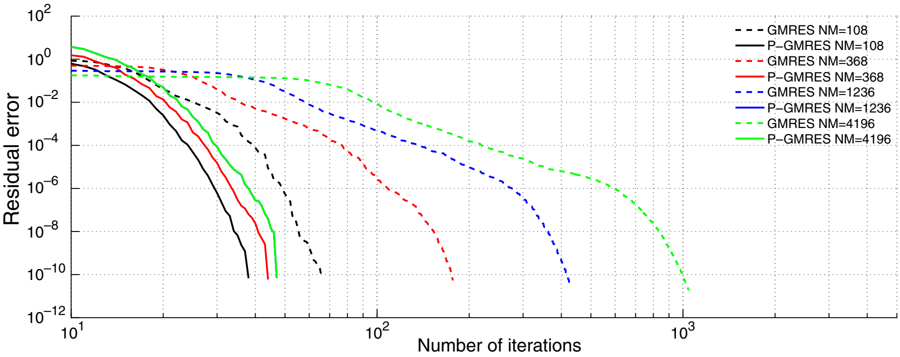


(b) Single Traces Formulation

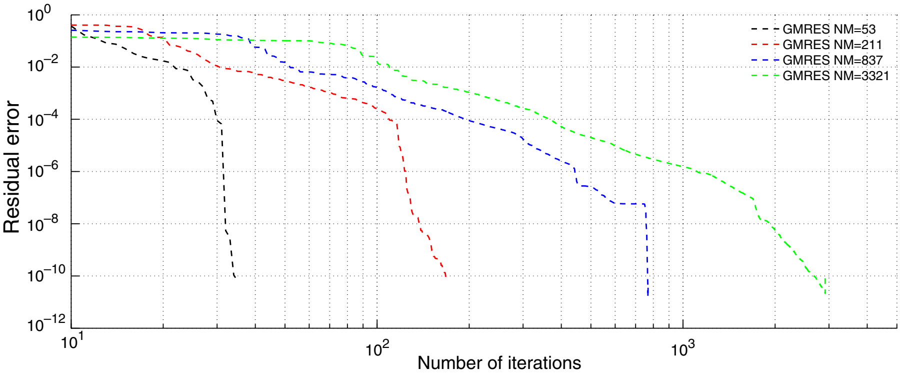
Fig. 11 MTF and STF local errors in different norms for the exterior Dirichlet and Neumann traces on a split circular scatterer of radius one ($N = 2$) with $\kappa_1 = \kappa_2 = 2$ and $\kappa_0 = 1$ for an impinging plane wave with angle $\theta = 0$. Observe the irregularity at triple points. Errors are obtained with respect to the linear interpolation of the exact Mie series solution

different boundary mesh refinements for both unconditioned and diagonally preconditioned GMRES. One can observe the large reduction in number of iterations almost independent on the number of variables and the comparison with STF in Fig. 12b. For example, in the case $\kappa_1 = 2$ and $\kappa_2 = 8$, the preconditioned MTF attains residual error tolerance at 44 and 47 iterations for $NM = 940$ and $NM = 4{,}188$, whereas the number of iterations for the non-preconditioned system increases from 369 to 1,077. A slight increase in number of iterations is observed for different wavenumbers κ_1 and κ_2. However, large-contrasting values were not modeled due to memory requirements.

Unfortunately, diagonal preconditioning requires the inversion of $\mathbf{Z}_{N,h}$ which increases computation time. Hence, Calderón preconditioning is



(a) Multiple Traces Formulation



(b) Single Traces Formulation

Fig. 12 Residual errors against number of iterations for GMRES using MTF (**a**) and STF (**b**), and diagonally preconditioned GMRES using MTF (**a**) (P-GMRES), for different numbers of degrees of freedom NM. Geometry considered is a circle of radius one divided in two halves with $\kappa_0 = 1$ and equal values $\kappa_1 = \kappa_2 = 2$

expected to deliver better results as it yields compactly perturbed block identities.

5.3.1 Matrix eigenvalues distribution

Eigenvalue distributions for the previous test geometries are presented in Fig. 14a and b. One observes that after diagonal preconditioning eigenvalues spread strongly with respect to the non-preconditioned system. These eigenvalues correspond to propagative modes. Also, an increase in the number of degrees of freedom translates into denser clusters. This is due to the resolution of higher frequencies. Still, the most interesting features of the MTF are to be found when applying Calderón preconditioning.

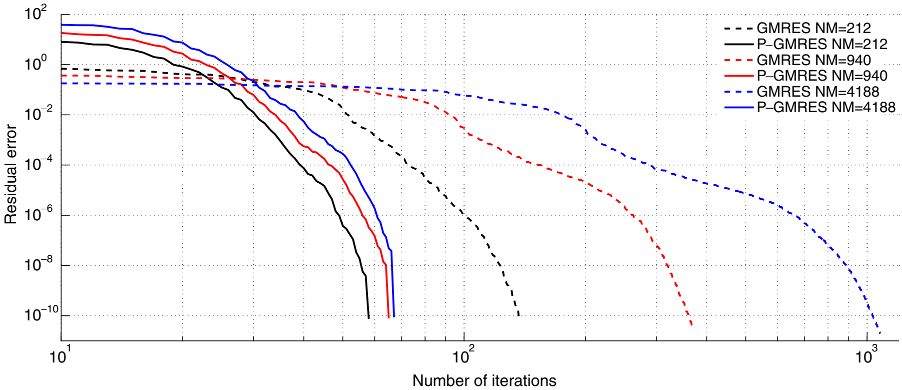
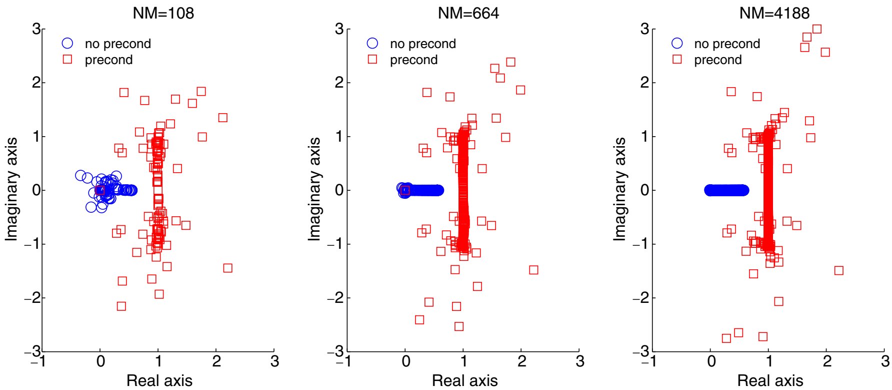
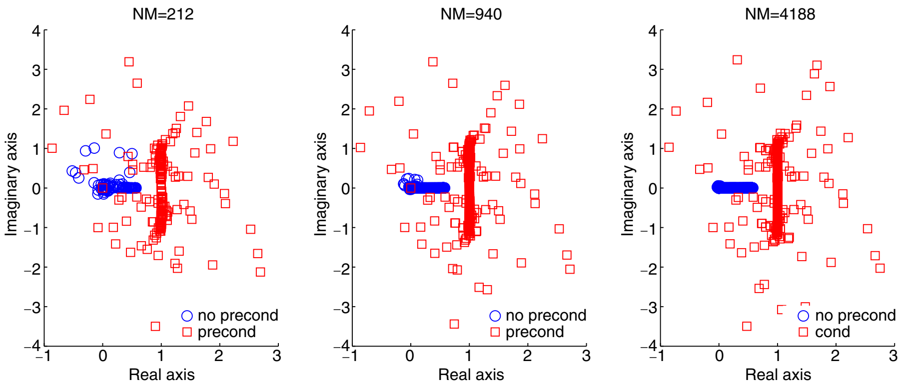


Fig. 13 Residual errors against number of iterations for GMRES and preconditioned GMRES (P-GMRES) when solving the multitrace formulation for two semicircles of radius one with wavenumbers $\kappa_0 = 1$, $\kappa_1 = 2$ and $\kappa_2 = 8$



(a) $\kappa_0 = 1$ and $\kappa_1 = \kappa_2 = 2$.



(b) $\kappa_0 = 1$, $\kappa_1 = 2$ and $\kappa_2 = 8$.

Fig. 14 Spectrum distributions for the matrix $\mathbf{M}_{2,h}$ and preconditioned system $\mathbf{Z}_{2,h}^{-1}\mathbf{M}_{2,h}$ for different values of $NM = \mathcal{O}(h^{-1})$. Geometry considers two semicircles of radius one

6 Conclusions and future work

We have introduced a stable formulation for the Helmholtz transmission problem over multiple domains. It ensures uniqueness of the solution so that the spurious mode problem is completely set aside. Moreover, its implementation is straightforward as it only requires standard discretization bases. Also, we have shown a great reduction in the number of iterations when block-diagonal preconditioning is used. However, inverting this matrix is quite consuming and therefore Calderón type preconditioning is currently explored to solve this issue. Future work includes extension to HTP with Dirichlet conditions or screens, to operators of div-grad form and, to vectorial electromagnetic scattering.

References

1. Ammari, H.: An Introduction to Mathematics of Emerging Biomedical Imaging. Mathematics & Applications, vol. 62. Springer, Berlin (2008)
2. Bony, J.M.: Cours d'analyse: Théorie des distributions et analyse de Fourier. Éditions de l'École Polytechnique, Palaiseau, France (2001)
3. Buffa, A., Hiptmair, R.: Regularized combined field integral equations. Numer. Math. **100**(1), 1–19 (2005)
4. Buffa, A., Hiptmair, R., von Petersdorff, T., Schwab, C.: Boundary element methods for Maxwell equations on Lipschitz domains. Numer. Math. **95**(3), 459–485 (2003)
5. Christiansen, S., Nédélec, J.C.: Des préconditionneurs pour la résolution numérique des équations intégrales de frontiére de l'acoustique. C. R. Acad. Sci. Paris, Ser. I Math **330**(7), 617–622 (2000)
6. Costabel, M.: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. **19**(3), 613–626 (1988)
7. Costabel, M., Dauge, M., Nicaise, S.: Singularities of Maxwell interface problems. M2AN **33**(3), 627–649 (1999)
8. Costabel, M., Stephan, E.: A direct bounary equation method for transmission problems. J. Math. Anal. Appl. **106**, 367–413 (1985)
9. Dauge, M.: Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988)
10. Farhat, C., Roux, F.X.: Implicit parallel processing in structural mechanics. Comput. Mech. Adv. **2**(1), 124 (1994)
11. Frühauf, F., Gebauer, B., Scherzer, O.: Detecting interfaces in a parabolic-elliptic problem from surface measurements. SIAM J. Numer. Anal. **45**(2), 810–836 (2007)
12. Gatica, G.N., Hsiao, G.C., Mellado, M.E.: A domain decomposition method based on BEM and FEM for linear exterior boundary value problems. J. Math. Anal. Appl. **262**(1), 70–86 (2001)
13. Haase, G., Heise, B., Kuhn, M., Langer, U.: Adaptive domain decomposition methods for finite and boundary element equations. In: Wendland, W. (ed.) Boundary Element Topics. Proceedings of the Final Conference of the Priority Research Programme "Boundary Element Methods" 1989–1995 of the German Research Foundation, 2–4 Oct 1995 in Stuttgart, Germany, pp. 121–147. Springer, Berlin (1997)
14. Hammer, M., Schweitzer, D., Michel, B., Thamm, E., Kolb, A.: Single scattering by red blood cells. Appl. Opt. **37**(31), 7410–7418 (1998)
15. Harrington, R.: Boundary integral formulations for homogeneous material bodies. J. Electromagn. Waves Appl. **3**(1), 1–15 (1989)
16. Hazard, C., Lenoir, M.: On the solution of time-harmonic scattering problems for Maxwell's equations. SIAM J. Math. Anal. **27**(6), 1597–1630 (1996)

17. Hiptmair, R.: Operator preconditioning. Comput. Math. Appl. **52**(5), 699–706 (2006)
18. Hiptmair, R., Jerez-Hanckes, C.: Multiple traces boundary integral formulation for Helmholtz transmission problems. Tech. Rep. 35, SAM ETH Zurich (2010)
19. Hiptmair, R., Meury, P.: Stabilized FEM-BEM coupling for Helmholtz transmission problems. SIAM J. Numer. Anal. **44**, 2107–2130 (2006)
20. Hsiao, G., Wendland, W.: Domain decomposition in boundary element methods. In: Glowinski, R., Kuznetsov, Y., Meurant, G., Périaux, J., Widlund, O. (eds.) Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, pp. 41–49. SIAM Philadelphia (1991)
21. Hsiao, G.C.: Boundary element methods—An overview. Appl. Numer. Math. **56**(10–11), 1356–1369 (2006)
22. Hsiao, G.C., Steinbach, O., Wendland, W.L.: Domain decomposition methods via boundary integral equations. J. Comp. Appl. Math. **125**(1–2), 521–537 (2000)
23. Hsiao, G.C., Wendland, W.L.: Boundary integral equations. Applied Mathematical Sciences, vol. 164. Springer, Berlin (2008)
24. Jerison, D., Kenig, C.: The inhomogeneous Dirichlet problem in Lipschitz domains. J. Funct. Anal. **130**, 161–219 (1995)
25. Kellogg, R.B.: Higher order singularities for interface problems. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972), pp. 589–602. Academic, New York (1972)
26. Kenig, C.E.: Carleman estimates, uniform Sobolev inequalities for second-order differential operators, and unique continuation theorems. In: Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986), vol. 1, pp. 948–960. Amer. Math. Soc., Providence, RI (1987)
27. Kondrat'ev, V.: Boundary value problems for elliptic equations in domains with conical or angular points. Trans. Moscow Math. Soc. **16**, 227–313 (1967)
28. Kreß, R., Roach, G.F.: Transmission problems for the Helmholtz equation. J. Math. Phys. **19**(6), 1433–1437 (1978)
29. Laliena, A.R., Rapún, M.L., Sayas, F.J.: Symmetric boundary integral formulations for Helmholtz transmission problems. Appl. Numer. Math. **59**(11), 2814–2823 (2009)
30. Langer, U., Steinbach, O.: Boundary element tearing and interconnecting methods. Computing **71**(3), 205–228 (2003)
31. Lions, J.L.: Équations différentielles opérationnelles et problèmes aux limites. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer, Berlin (1961)
32. Lions, J.L., Magenes, E.: Problèmes aux Limites Non Homogènes, vol. 1. Dunod, Paris, France (1968)
33. Mardal, K., Winther, R.: Preconditioning discretizations of systems of partial differential equations. Numer. Linear. Algebr. **18**(1), 1–40 (2010)
34. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge, UK (2000)
35. McLean, W., Steinbach, O.: Boundary element preconditioners for a hypersingular integral equation on an interval. Adv. Comput. Math. **11**(4), 271–286 (1999)
36. Meury, P.: Stable finite element boundary element Galerkin schemes for acoustic and electromagnetic scattering. ETH Dissertation No. 17320, ETH Zürich, Zürich, Switzerland (2007)
37. Müller, C.: On the behavior of the solutions of the differential equation $\Delta U = F(x, U)$ in the neighborhood of a point. Comm. Pure Appl. Math. **7**, 505–515 (1954)
38. Nazarov, S., Plamenevskii, B.: Elliptic Problems in Domains with Piecewise Smooth Boundaries, Expositions in Mathematics, vol. 13. Walter de Gruyter, Berlin (1994)
39. Nédélec, J.C.: Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems. Applied Mathematical Sciences, vol. 44. Springer, Berlin (2001)
40. Nicaise, S., Sändig, A.M.: General interface problems I/II. Math. Methods Appl. Sci. **17**, 395–450 (1994)
41. Quarteroni, A., Sacco, R., Saleri, F.: Numerical Mathematics. Texts in Applied Mathematics, vol. 37. Springer, New York (2000)
42. Sadiku, M.: Numerical Techniques in Electromagnetics, 2nd edn. CRC Press LLC, New York (1996)

43. Sauter, S., Schwab, C.: Boundary Element Methods, Springer Series in Computational Mathematics, vol. 39. Springer, Heidelberg (2010)
44. Schatz, A.: An observation concerning Ritz–Galerkin methods with indefinite bilinear forms. Math. Comput. **28**, 959–962 (1974)
45. Sommerfeld, A.: Partial Differential Equations in Physics. Pure and Applied Mathematics. Academic, New York (1949)
46. Steinbach, O.: Numerical approximation methods for elliptic boundary value problems. Springer, New York (2008)
47. Steinbach, O., Wendland, W.: The construction of some efficient preconditioners in the boundary element method. Adv. Comput. Math. **9**, 191–216 (1998)
48. Steinbach, O., Windisch, M.: Robust boundary element domain decomposition solvers in acoustics. Tech. Rep. 9, Institut für Numerische Mathematik, TU Graz (2009)
49. von Petersdorff, T.: Boundary integral equations for mixed Dirichlet, Neumann and transmission problems. Math. Methods Appl. Sci. **11**, 185–213 (1989)
50. Warnick, K.: Numerical Analysis for Electromagnetic Integral Equations. Artech House, Norwood, MA (2008)
51. Webb, A.: Introduction to Biomedical Imaging. Wiley, New York (2003)